Virginia Tech Regional Math Contest: Saturday, Oct. 25. Location: Himes Hall, LSU. You **must** be present promptly at 8:30 A.M., or you will not be able to enter the building.

Problem Solving Seminar - Fall 2014
Oct. 15

Useful facts:

- The prime factorization of the current year is $2014 = 2 \cdot 19 \cdot 53$, and the previous year was $2013 = 3 \cdot 11 \cdot 67$.
- **Fermat's Little Theorem.** If p is prime and a is any integer, then $a^p - a$ is a multiple of p.
- For any integer n, its square can only have the following remainders:

  0 or 1 when divided by 3; 0 or 1 when divided by 4; 0, 1, 4, 5, or 9 when divided by 10.

- Remember that calculators are **not** allowed on the Virginia Tech Contest or Putnam Exam!

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1. Observe that $49^2 = 2401$, and so $49^2 - 1 = 2400$, which ends with two zeros.
   (a) Are there other integers n such that $n^2 - 1$ ends with two zeros?
   (b) Given any $k \geq 1$, is there an integer n such that $n^2 - 1$ ends with k zeros?
2. (a) Find the positive integer solutions (x, y, z) to
   $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$
   (b) Does your answer to (a) change if the solutions are allowed to include negative integers?
   (c) Find all integer solutions to
   $$\frac{1}{x} + \frac{1}{y} = \frac{1}{6}.$$
   (d) If p is prime, how many integer solutions (x, y) are there to
   $$\frac{1}{x} + \frac{1}{y} = \frac{1}{p}?$$
3. (a) What is the last digit of $3^{3^{3^3}}$?
   (b) What are the last two digits of 2014^{2013}?
4. (a) What is the remainder when 2014^{2012} is divided by 2013?
   (b) What is the remainder when 2012^{2014} is divided by 2013?
   (c) Prove that $105^{105} + 213^{213}$ is a multiple of 2014.
   Hint: This is the same as saying that the expression is a multiple of each divisor of 2014 *– use the prime factorization.*
5. Are there any perfect squares whose only digits are 0s and 6s?

6. (a) Show that if there is an integer solution to $a^2 + b^2 = c^2$, then at least one of a and b is a multiple of 3.
   (b) Show that there are no integer solutions to
   $$3x^2 + 7y^2 = 4z^2 - 3.$$
   Hint: Consider the remainder when divided by 4.
   (c) [Gelca-Andreescu **706**] Show that there are no positive integer solutions to
   $$x^2 + 10y^2 = 3z^2.$$
   Hint: It is helpful to consider the equation modulo 3 *or* 10.

7. [VTRMC **1979 #5**] Show, for all positive integers $n = 1, 2, \ldots$, that 14 divides $3^{4n+2} + 5^{2n+1}$.

8. [Putnam **1979 A1**] Find the set of positive integers with sum 1979 and maximum possible product.

   Extra challenge: Can you also answer this question for 2014*?*

9. (a) Prove that if n is odd, then $10^{10^n} + 10^n$ is a multiple of 11.
   (b) Prove that $10^{10^{10}} + 10^{10}$ is **not** a multiple of 11, but it is a multiple of 101.

10. [Putnam **2010 A4**] Prove that for each positive integer n, the number
   $$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$
   is not prime.

Challenge.

1. (a) In a certain country postage stamps are sold in two denominations: 5 cents and 12 cents. The first-class postage rate has risen in recent years from 25 to 29 cents, then 34, and finally 37 cents. Show how each of these values can be exactly achieved with some combination of stamps.
   (b) The most recent proposal called for a rise to 43 cents until an observant mathematician pointed out that this value was impossible to achieve! The new rate was instead set for 42 cents. Prove the claim, that 43 cents is impossible.
   (c) Furthermore, prove that 43 cents is the largest impossible value, so that any rate 44 cents or higher **can** be achieved.
   (d) Now consider the general problem: the postage stamps are available in a cents and b cents, where a and b have no common divisors (why?). What is the largest postage rate that cannot be made exactly with the available stamps? Prove that all larger values can be achieved.