MATH 7230 Homework 2 - Spring 2014

Due Thursday, Feb. 20 at 1:30

The notation "I-K" is shorthand for the textbook.

1. Using partial summation as in lecture, fill in the details of the proof that $\theta(x) \sim x$ implies that $\pi(x) \sim \frac{x}{\log x}$.

   Hint: You may find it more convenient to use PNT in the form $\pi(x) \sim \mathrm{li}(x)$.

2. Use partial summation to verify that the following two versions of Selberg's asymptotic formula are equivalent:

$$\psi(x)\log(x) + \sum_{n \le x} \Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\log x + O(x),$$

$$\sum_{n \le x} \Lambda(n)\log n + \sum_{jk \le x} \Lambda(j)\Lambda(k) = 2x\log x + O(x).$$

(Optional) The Euler summation formula states that (for $n \in \mathbb{N}$)

$$\sum_{1 < n \le x} f(n) = \int_1^x f(x)dx + \int_1^x \{x\}f'(x)dx.$$

Such sums are also often evaluated using partial summation, with $g(x) \equiv 1$.

(a) Can you find an example where this gives a weaker result than summation by parts?

(b) Can you find example where this gives a stronger result than summation by parts?

3. The left-hand side of the second version of Selberg's asymptotic formula is sometimes written as $\sum_{n \le x} \Lambda_2(n)$, where

$$\Lambda_2 := \log \cdot \Lambda + \Lambda \star \Lambda.$$

(a) Referring to I-K equation (1.43), prove that this function is equivalent to

$$\Lambda_2 = \mu \star \log^2.$$

Hint: To compare the formulas explicitly, recall that $\Lambda(n)$ *is* 0 *unless* n *is a prime power.*

(Optional) Prove (1.44) – (1.46), and also the formula in between (1.46) and (1.47) on p. 16. Furthermore, complete the exercise on p. 39. Note that the recurrence it refers to is (1.44).

4. In the proof of Selberg's asymptotic formula we used the bound ($m \ge 1$)

$$\sum_{k \le x} \left(\log \frac{x}{k}\right)^m = O(x).$$

The proof from lecture used the seemingly "rough" bound

$$\left(\log \frac{x}{k}\right)^m = O(x^\delta),$$

which holds for any $0 < \delta < 1$. An integral comparison then gives the bound $O(x)$ for the sum.

(a) Begin by proving that

$$\sum_{k \le x} (\log k)^m = xP_m(\log x) + O\left((\log x)^m\right)$$

for certain polynomials $P_m(y)$. Use iterated integration by parts to find these polynomials.

(b) Is the bound from lecture sharp? In other words, is it true that

$$\sum_{k \le x} \left(\log \frac{x}{k}\right)^m \gg x?$$

(c) Use partial summation to evaluate the sum for $m = 1, 2$ and 3 (for $m = 1$ you should recover Stirling's formula). Is there a general formula for arbitrary m? *Hint: Refer to I-K* (1.63) – (1.64) *after you have tried to solve this problem on your own.*

(Optional) How do these sums compare to $\sum_{n \le x} \Lambda_k(n)$? Recall from (1.45) that $\Lambda_k(n) = O\left((\log n)^k\right)$.

5. Prove that

$$\sum_{n \le x} \frac{\mu(n)}{n} = O(1).$$

Compare to the sums

$$\sum_{n \le x} \frac{1}{n} \quad \text{and} \quad \sum_{n \le x} \frac{(-1)^n}{n}.$$

What does this imply about $\mu(n)$?

Remark: Actually, even more is true: $\left|\sum_{n \le x} \frac{\mu(n)}{n}\right| < 1$. *See I-K (2.18), but first try it on your own.*

6. As mentioned in lecture, a useful bound is

$$\sum_{jk \le x} \frac{\Lambda(j)\Lambda(k)}{\log(jk)} \ll x.$$

In this problem you will prove a simpler version of this bound, namely

$$\sum_{pq \le x} \frac{\log(p)\log(q)}{\log(pq)} \ll x,$$

where p and q are primes.

(a) Use the Arithmetic-Geometric Mean inequality to show that

$$\log(p)\log(q) \leq \sqrt{\frac{\log p + \log q}{2}}.$$

(b) Now estimate the sum

$$\sum_{pq \leq x} \frac{1}{\sqrt{\log(pq)}}$$

to prove the overall bound.

(c) Is the bound sharp? In other words, is the asymptotic order x?

(Optional) Can you also prove the bound with $\Lambda(j), \Lambda(k)$?