MATH 7230 Homework 4 - Spring 2014

Due Thursday, Mar. 27 at 1:30

1. If $A \subset B \subset \mathbb{N}$, then the *arithmetic density* of A in B is

$$d_B(A) := \lim_{X\to\infty} \frac{\#\{n \le X \mid n \in A\}}{\#\{n \le X \mid n \in B\}}.$$

Similarly, assuming that B is *substantial* (so that $\sum_{n\in B} \frac{1}{n} = \infty$), the *logarithmic (Dirichlet) density* is

$$D_B(A) := \lim_{s\to 1^+} \frac{\displaystyle\sum_{n\in A} \frac{1}{n^s}}{\displaystyle\sum_{n\in B} \frac{1}{n^s}}.$$

In this problem you will compare the two notions of density. If the type of density is not specified, then the statement applies to both.

(a) Prove that if A is finite, then the density of A is zero (relative to any infinite set).

(b) Prove that if A_1, A_2 are disjoint and have densities δ_1, δ_2, respectively, then $A_1 \cup A_2$ has density $\delta_1 + \delta_2$.

(c) Use (a) and (b) to conclude that if $A_1 \cup A_2$ has density greater than 1, then $A_1 \cap A_2$ is infinite.

(Optional) Let α, β be positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and set

$$A := \Big\{ \lfloor n\alpha \rfloor \ \mid n \ \in \mathbb{N} \Big\},$$
$$B := \Big\{ \lfloor n\beta \rfloor \ \mid n \ \in \mathbb{N} \Big\}.$$

Prove that A has density $\frac{1}{\alpha}$ and B has density $\frac{1}{\beta}$. In fact, Beatty's Theorem states that A and B are disjoint (try and prove it!).

As an application, write down powers of 10 in two lists, one in base 2 and one in base 5:

$$1_2, 1010_2, 1100100_2, \ldots$$
$$1_5, 20_5, 400_5, \ldots.$$

Conclude that for $n \ge 2$ there is an entry with n digits in exactly one of the two lists.

2. (a) Let A be the set of positive integers whose first digit is 1. Fill in the details of the claims made in class:

   (i) The arithmetic density of A does not exist;

   (ii) The logarithmic density of A is $\log_{10} 2 \cong 0.301$.

   (b) Prove that if $d_B(A) = \delta$, then the logarithmic density is also $D_B(A) = \delta$.

   Hint: Use partial summation.

3. You are encouraged to read the following sources on the Gamma function if you are unfamiliar with its basic properties.

   A thorough introduction is found in this Second-Year Essay (author unknown):

   `http://warwickmaths.org/files/gamma.pdf`

   Quick proofs of the reflection and duplication formulas are on Pages 8 – 9 of R. Koekoek's notes:

   `http://aw.twi.tudelft.nl/~koekoek/documents/wi4006/gammabeta.pdf`

   A concise summary of Weierstrass products is found in Section 1 of M. Nica's notes:

   `http://math.nyu.edu/~nica/last_complex.pdf`

4. In this problem you will fill in the details for the error terms in Laplace's method for asymptotic expansions applied to the Gamma function. Recall that we used simple changes to variables to write the integral as

$$\Gamma(s+1) = s^{s+1}e^{-s}\int_0^\infty e^{-s(u-1)+s\log u}du.$$

   Denote the integral by $I(s) := \displaystyle\int_0^\infty e^{s\cdot g(u)}du$, where $g(u) := -(u-1) + \log(1+(u-1))$.

   (a) For $1 > \varepsilon > 0$, set $I_1(s) := \displaystyle\int_{1-\varepsilon}^{1+\varepsilon} e^{s\cdot g(u)}du$, so that

$$I(s) - I_1(s) = E_1^-(s) + E_1^+(s) := \int_0^{1-\varepsilon} + \int_{1+\varepsilon}^\infty.$$

   Prove that the error terms are exponentially small compared to $I(s)$, i.e., that $E_1^\pm(s) = O\left(e^{-cs}\cdot I(s)\right)$ for some constant c.
   Hint: For E_1^-, show that $g(u)$ is increasing on $(0,1)$. For E_1^+, show that $g(u) \le -\frac{\varepsilon}{2}(u-1)$ for $u \ge 1+\varepsilon$.

   (b) Let

$$I_2(s) := \int_{1-\varepsilon}^{1+\varepsilon} e^{-s\frac{(u-1)^2}{2}}du, \tag{1}$$

   and use Taylor's Theorem (with remainder) on e^x to show that

$$I_1(s) - I_2(s) \le \int_{1-\varepsilon}^{1+\varepsilon} e^{-s\frac{(u-1)^2}{2}}\cdot e^{c(u)}s\cdot\left|g(u) - \frac{(u-1)^2}{2}\right|du,$$

   where $c(u) \in \left[0, s\left|g(u) - \frac{(u-1)^2}{2}\right|\right]$. Now apply Taylor's Theorem to $g(u)$ (around $u = 1$) and conclude that

$$\left|g(u) - \frac{(u-1)^2}{2}\right| \le C\cdot(u-1)^3,$$

for some constant C. Furthermore, for sufficiently small ε, $c(u) \leq \frac{s(u-1)^2}{4}$. Use these bounds to verify that

$$I_1(s) - I_2(s) \ll \frac{1}{s}.$$

(c) Finally, set $I_3(s) := \int_{-\infty}^{\infty} e^{-s\frac{u^2}{2}}\,du$. Show that

$$I_2(s) - I_3(s) = 2\int_{\varepsilon}^{\infty} e^{-\frac{su^2}{2}}\,du \ll \frac{1}{\sqrt{s}} e^{\frac{s\varepsilon^2}{2}}$$

by making the change of variables $u \mapsto u-\varepsilon$ and then comparing to the Gamma function. Conclude that $I(s) \sim I_3(s)$.

5. The *exponential integral* is defined for positive x by

$$E_1(x) := \int_x^{\infty} e^{-t}\frac{dt}{t}.$$

Prove that it has an asymptotic expansion as $x \to \infty$ given by

$$E_1(x) \sim \frac{e^{-x}}{x} \sum_{n\geq 0} \frac{(-1)^n n!}{x^n}.$$

Remark. In 1963 Sweeney used the above asymptotic expansion along with the identity

$$E(x) = -\log x - \gamma - \sum_{n\geq 1} \frac{(-x)^n}{n \cdot n!}$$

to calculate several thousand digits of Euler's constant. This identity is best proven using the Weierstrass product for $\Gamma(s)$; see Section 2.3 of `http://numbers.computation.free.fr/Constants/Gamma/`

6. Recall that the *error function* is given by $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$.

(a) We showed that the Taylor expansion is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{pi}} \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!}.$$

Although this series converges absolutely for all x, it does so slowly for large x. In particular, prove that the largest absolute value of the summands occurs when $n \asymp x^2$. *Hint: Apply Stirling's approximation and then take the derivative with respect to n.*

(b) We also showed the asymptotic expansion

$$\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n\geq 0} \frac{(-1)^n (2n)!}{2^{2n} n! x^{2n+1}}.$$

Prove that this diverges for all $x \neq 0$.