MATH 7230 Homework 5 - Spring 2014

Due Thursday, May 1 at 1:30

1. Recall that the Mellin transform of a function ϕ is

$$\widetilde{\phi}(s) := \int_0^\infty \phi(t) t^{s-1} dt.$$

Prove that if $g(t) = \sum_{m\geq 1} f(mt)$, then

$$\widetilde{g}(s) = \zeta(s) \cdot \widetilde{f}(s).$$

(Optional): *Describe the domain in which g is meromorphic (or has a meromorphic) continuation, given such a domain for f. For example, f might be analytic in $a < Re(s) < b$.*

2. I-K Exercises 4.1.1 & 4.1.2.

Note that although the textbook defines the Bernoulli polynomials in terms of the recurrences (4.8) – (4.10), you may alternatively take the generating function (4.11) as the definition of $B_k(X)$.

(Optional) : *Fill in the details of the proof of the summation formula*

$$\sum_{n=1}^{m} n^k = \frac{1}{k+1}\Big(B_{k+1}(n+1) - B_{k+1}\Big).$$

3. For a positive integer k (or even a positive real number), define $F_k(t) := \sum_{n\geq 1} e^{-n^k t}$. Recall from lecture that we used Euler-Maclaurin summation and Mellin transform to prove the asymptotic expansions (as $t \to 0$)

$$F_1(t) \approx \frac{1}{t} - \frac{1}{2} + O(t),$$
$$F_2(t) \approx \frac{\sqrt{\pi}}{\sqrt{t}} + O\left(t^N\right) \quad \text{for any } N.$$

Prove the most precise statement that you can for general k.

(Optional): *Do the same for $F_{k,\ell}(t) := \sum_{n\geq 1} n^\ell e^{-n^k t}$, where $\ell \in \mathbb{R}$.*

4. Recall that Bonferroni's inequalities state that if $n \in \mathbb{N}$ and $k \geq 0$, then

$$\sum_{\substack{d|n \\ \omega(d)\leq k}} \mu(d) \qquad \text{is} \begin{cases} \geq 0 & \text{if } k \text{ even,} \\ \leq 0 & \text{if } k \text{ odd.} \end{cases}$$

(a) If $n = p_1^{a_1} \cdots p_r^{a_r}$, rewrite the sum in terms of binomial coefficients.

(b) Prove that

$$\sum_{i=0}^{m}(-1)^i\binom{\ell}{i} = (-1)^m\binom{\ell-1}{m}.$$

Hint: Three possible approaches: Induction; combinatorial arguments; generating functions.

(c) Use Stirling's formula to give rough estimates for $\binom{n}{m}$ if both n and m are large (and $m < n$). Combined with parts (a) and (b), this gives bounds for the number of error terms in Brun's combinatorial sieve.

5. Recall that the basic goal in sieving is to find good estimates of

$$S_a(x,z) = \sum_{\substack{n\le x \\ p\nmid n\ \forall\ p\le z}} a_n.$$

A key technical point in many sieve results (including the so-called Fundamental Lemma) is the concept of *sifting dimension*. As in I-K equation (6.34), this quantity is any $\kappa > 0$ such that

$$\prod_{w\le p<z}\frac{1}{1-g(p)} \le K\cdot\left(\frac{\log z}{\log w}\right)^{\kappa}, \tag{1}$$

where K is a constant that does not depend on w. Recall that $g(p)$ is the "local density" of the sieved set when restricted to multiples of p. Find κ for the following cases:

(a) a_n is the indicator function for an arithmetic progression $b \bmod q$, with $q \le z$. Note that this is a good sieve to use for studying Dirichlet's theorem on primes in arithmetic progressions.

(b) a_n is the indicator for the set of integers $\{n = m(m+2)\ :\ m\in\mathbb{N}\}$. This is a natural sieve for studying Twin Primes

(Optional): *In fact, the sifting dimension can also be compared to Mertens' weighted prime counting function. In particular, if*

$$\sum_{p\le x} g(p)\log p \sim \kappa\log x, \tag{2}$$

then on comparing to Merten's result that $\sum_{n\le x}\frac{\Lambda(n)}{n} \sim \log x$, *we see that* $g(p) \sim \frac{\kappa}{p}$ *on "average".*

Show that (2) *implies* (1).

6. Recall that the generating function for Euler's partition function is

$$P(q) := \sum_{n\ge 0} p(n)q^n = \prod_{m\ge 1}\frac{1}{1-q^m}.$$

Consider the singularity near $q = 1$ and that as $t\to 0^+$,

$$\log P\left(e^{-t}\right) \sim \frac{\pi^2}{6t}.$$

(Optional): *Use the Euler-MacLaurin summation and Mellin transform technique to prove*

$$P\left(e^{-t}\right) \approx \sqrt{\frac{t}{2\pi}} e^{\frac{\pi^2}{6t} - \frac{t}{24}};$$

in other words, the error is $O\left(t^N\right)$ *for any* $N \geq 1$.

Refer to Zagier's survey as needed, particularly equation (46):

`http://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`