LSU Problem Solving Seminar - Fall 2015
Aug. 26

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/∼mahlburg/teaching/2015-Putnam.html`

Warm Up

---

1. You enter a classroom where a partially erased equation was left on the board:
$$2 \quad \_\_ \quad 2 = 4.$$
Note that this equation is true with the symbols "+" and "×". Are there any other integers a, b and c that also have this property: namely, that
$$a \quad \_\_ \quad b = c$$
holds for both addition and multiplication?

2. Alex normally completes a 10 mile bike ride along the flat river path in one hour, but today she plans to climb a 5-mile long hill and then return. She reasons: "I will only be able to manage 8 miles an hour on the way up, but I will speed up to 12 miles an hour on the way down, so this should work out to an average of 10 miles an hour." How long does her ride actually take?

3. A man is staying in a hotel with 11 floors. Beginning from the first floor, he climbs all the way to the 11th and descends to the 2nd floor. He then climbs back to the 11th floor, and descends to the 3rd floor. He continues like this until he has descended to the 10th floor and climbs back to the 11th floor a final time to rest.
   (a) How many flights of stairs did he traverse, including both up and down? Note that the first ascent is **10** flights.
   (b) How many flights would he traverse if he did the same thing in a hotel with n floors?

Main Problems

---

4. (a) [Gelca-Andreescu **11**] For a positive integer n, define
$$R(n) := 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n},$$
$$S(n) := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$
Calculate the first several values of each series. Conjecture a relationship between $R(n)$ and $S(n)$, and prove it.

(b) Prove that for $n \geq 2$,
$$\frac{1}{2} < S(n) < 1.$$

(c) Evaluate $\lim_{n\to\infty} R(n)$ and $\lim_{n\to\infty} S(n)$.

5. A collection of distinct lines are drawn in the plane, separating it into distinct regions. For example, one line creates 2 regions, and two parallel lines create 3 regions, although two intersecting lines create 4 regions.

   (a) What is the maximum number of regions that can be created with 3 lines? With 4 lines?

   (b) What is the maximum number of regions that can be created with n lines?

   (c) If a circular pie is divided by n straight knife cuts, what is the maximum number of pieces? The pieces do not need to be the same shape or size.

   (d) Suppose that n points are placed on the boundary of a circle, and all possible lines between pairs of points are drawn, separating the circle into various regions. What is the maximum number of regions that can be formed with n points?

   Hint: The values for small n might be misleading...

6. Prove that if n and k are positive integers, then
$$\frac{n^{k+1}}{k+1} < 1^k + 2^k + 3^k + \cdots + n^k < \frac{(n+1)^{k+1}}{k+1}.$$

7. [Putnam **2004 A1**] Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?