LSU Problem Solving Seminar - Fall 2015
Oct. 28: Probability

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Website: `www.math.lsu.edu/~mahlburg/teaching/2015-Putnam.html`

Warm Up:

1. At LSU, 92% of students like Mike the Tiger, 77% like Game Day, and 61% like Live Oak trees*. Prove that there is at least one student who likes all three campus sights.

   * Statistics may not be accurate.

2. Suppose that 4 dice are rolled.
   (a) What is the probability that the sum is odd?
   (b) What is the probability that the sum is a multiple of 3?
   Hint: If the dice were rolled one at a time, think about what must happen with the last die.
   (c) What is the probability that the product of the dice is odd?
   (d) What is the probability that the sum and product are both odd?

3. A famous probability "paradox" (which ultimately comes down to precise phrasing/definitions) is the Two Child problem.
   (a) Mr. Jones has two children, and tells you "my older child is a boy." What is the probability that both of his children are boys?
   (b) Mr. Smith has two children, and tells you "at least one of my children is a boy." What is the probability that both of his children are boys?
   Hint: Among the implict assumptions being made are that boys and girls are equally likely at birth, and that Mr. Smith is a random representative of **all** *families with two children and at least one boy.*

Main Problems:

4. A kitchen floor is tiled by square tiles that measure 12 inches per side. A buttered bagel is dropped on the floor, face-down; the bagel measures 6 inches in diameter.
   (a) What is the probability that the bagel lands entirely on one tile?
   (b) What is the probability that the bagel lands on a corner where 4 tiles meet?
   Remark: One of your answers should suggest an experimental method of approximating the value of π.

5. If the digits $0, 1, 2, \ldots, 9$ are written in a random order (i.e., a *permutation*), what is the probability that 1 and 2 are adjacent?

6. [Andreescu-Gelca **915**.] An exam consists of 3 problems selected randomly from a list of $2n$ problems, where $n \geq 1$. For a student to pass, he needs to solve at least two of the three problems. If a certain student can solve exactly half of the problems on the list, find the probability that he will pass the exam.

7. Three points are chosen at random on a circle of radius 1. These points determine three random arcs along the circle, with total length 2π (the entire diameter).

   (a) Prove that the expected (i.e. "average") length of each arc is $\frac{2\pi}{3}$.

   (b) Now mark a fixed base point P on the circle **before** the three random points are chosen. Prove that the expected length of the arc containing P is larger than $\frac{2\pi}{3}$!

8. [Putnam **2005 A6**] For $n \geq 4$, suppose that $P_1, \ldots, P_n$ are randomly, independently, and uniformly points chosen on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

Challenge:

9. [Related to VTRMC **2015 # 2**]. Consider a regular tetrahedron (four equilateral triangles as faces) with side length a.

   (a) Pick a coordinate system so that one of the faces is lying in the xy-plane, with vertices $(0,0), (a,0), \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$. Find the coordinates of the fourth vertex.

   (b) Prove that the volume of the tetrahedron is $\dfrac{a^3}{6\sqrt{2}}$.

   (c) Show that the *edge-angle* of the tetrahedron is $\arccos\left(\frac{1}{3}\right) \sim 70.53°$. Conclude that it is impossible to tile space with regular tetrahedra!