LSU Problem Solving Seminar - Fall 2015
Nov. 11: Functions

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Warm Up:

1. Let $A := \{a, b, c, d, e\}$ and $B := \{1, 2, 3\}$.
   (a) How many functions are there from A to B?
   (b) How many functions are there from B to A?
   (c) How many injective (1-to-1) functions are there from B to A?
   (d) How many invertible functions are there from A to B?

2. The *natural numbers* are $\mathbb{N} := \{1, 2, \ldots\}$. The integers are $\mathbb{Z} := \{\ldots, -1, 0, 1, 2, \ldots\}$.
   (a) Find all invertible functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is a multiple of n for all n.
   Hint: Suppose that $f(n) > n$ for some n, for example, $f(10) = 20$. What can you say about $f(5), f(2), f(1)$?
   (b) Find all invertible functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n)$ is a multiple of n for all n.

3. (a) Find all real functions f such that
$$f(x + y) = f(x) + y$$
   for all $x, y \in \mathbb{R}$.
   (b) Find all continuous real functions f such that
$$f(x + y) = f(x) + f(y)$$
   for all $x, y \in \mathbb{R}$.
   Remark: The theory of Hamel bases *implies that there are other discontinuous solutions!*

Main Problems:

4. [VTRMC **1990 # 3**] Let f be defined on the natural numbers as follows: $f(1) = 1$, and for $n > 1$,
$$f(n) = f(f(n - 1)) + f(n - f(n - 1)).$$
   Find, with proof, a simple explicit formula for $f(n)$ that is valid for all n.

5. (a) Find all continuous real functions f that satisfy the functional equation $f(f(x)) = x$ for all x.
   Hint: Observe that f is invertible, and is in fact its own inverse. Use the fact that an invertible continuous function is strictly increasing or decreasing. If f is increasing, is it ever possible that $f(x) > x$?

(b) Prove that there are no continuous real functions such that $f(f(x)) = -x$ for all x.

Hint: First consider $f(f(f(f(x))))$ to show that f is invertible.

6. Suppose that $k \geq 1$.

   (a) Prove that there is a unique continuous function $f : [0,1] \to [0,1]$ such that $f(0) = 1, f(1) = 0$ and

   $$f(x)^k - f(x)^{k+1} = x^k - x^{k+1} \qquad \text{for all } 0 \leq x \leq 1.$$

   Hint: Note that $f(x) = x$ is a solution to the functional equation, but it does not have the correct boundary values. Show that if $x \in [0,1]$ and $x^k - x^{k+1} = c$, then there is a second solution $z \in [0,1]$ satisfying $z^k - z^{k+1} = c$. Setting $f(x) = z$ is the correct choice.

   (b) Denote the function defined above by $f_k(x)$. Show that $f_1(x) = 1 - x$, and find a formula for $f_2(x)$.

   (c) Each $f_k(x)$ intersects the line $y = x$ in a unique point; find the value x_k such that $f_k(x_k) = x_k$.

7. [Andreescu-Gelca **8**] Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

   $$xf(y) + yf(x) = (x+y)f(x^2 + y^2).$$

   Hint: Suppose that $f(y) > f(x)$. Show that $(x+y)f(x) < xf(y) + yf(x) < (x+y)f(y)$.

8. [Putnam **2012 B1**] Let S be the set of functions from $[0, \infty)$ to $[0, \infty)$ with the following properties:

   (a) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S.

   (b) If $f(x)$ and $g(x)$ are in S, then $f(x) + g(x)$ and $f(g(x))$ are in S.

   (c) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then $f(x) - g(x)$ is in S.

   Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

9. [Putnam **1991 B2**] Suppose that f and g are real, non-constant, differentiable functions such that $f'(0) = 0$ and

   $$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + f(y)g(x) \end{aligned}$$

   for all x, y. Prove that $f(x)^2 + g(x)^2 = 1$ for all x.