Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 25 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Dec. 2.
- Putnam Mathematical Competition, **Sat., Dec. 5.** The Exam will take place in Lockett 244 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2015
Nov. 18: Miscellaneous Topics

Prof. Karl Mahlburg

Website: `www.math.lsu.edu/~mahlburg/teaching/2015-Putnam.html`

Warm Up:

1. A popular style of arithmetic puzzle is a *Cryptarithm*, where each letter is replaced by a digit so that the resulting expression is true.

$$\text{(a)}\quad \begin{array}{rcccccc} & N & U & M & B & E & R \\ + & N & U & M & B & E & R \\ \hline & P & U & Z & Z & L & E \end{array} \qquad\qquad \text{(b)}\quad \begin{array}{rccccc} & A & P & P & L & E \\ + & L & E & M & O & N \\ \hline B & A & N & A & N & A \end{array}$$

See `Cryptarithms.com` *and* `http://www.iread.it/cryptarithms.php`.

2. (a) Consider a chess board – an 8×8 grid of squares. If a straight line is drawn between opposite corners, how many of the small squares does the line cross? Corners do not count; the line *crosses* a square only if it passes through some part of its interior.
   (b) If the board is 4×8, how many squares does the diagonal line cross? If the board is 4×6?
   (c) (Harder!) Solve the general problem: How many squares does the diagonal line on an $m \times n$ grid cross?

3. Two players compete in a game that begins with 40 coins on the table. The players alternate turns, and can remove $2, 3$ or 4 coins each time. The winner is the player who **takes** the last coin. Prove that the first player can always win, and describe his winning strategy.

   Hint: What if the game began with only 10 *coins?*

Main Problems:

4. A dinner party is held with 9 people who are assigned seats around a circular table, with Person 1 assigned to Seat 1, followed by Person 2 in Seat 2, and so on up to Seat 9, in clockwise order.

(a) Later in the evening after dinner, the people sit back down disregarding the assigned seats. One of the guests notices that by coincidence they have now completely reversed order, so that now Person 9 is followed by Person 8, down to Person 1, in clockwise order. Another guest then comments: “If we preserve this order, then no matter how we rotate, exactly **one** of us will be in the correct seat!” Prove this claim.

(b) Even later in the evening, they sit down again at random, this time with no one in the assigned seat. Prove that there is some rotation such that **two** of them will be in the correct place.

5. Suppose that A is an $n \times n$ matrix with integer coefficients.

   (a) If every entry in A is even, prove that $\det(A)$ is a multiple of 2^n.

   (b) If every entry in A is odd, prove that $\det(A)$ is a multiple of 2^{n-1}.

   Hint: What is the effect of applying the elementary matrix operation that subtracts the first row from the second row?

6. (a) Find all real numbers x such that $(x-1)^4 = x^4$.

   (b) Find all complex numbers z such that $(z-1)^4 = z^4$.

7. [Andreescu-Gelca **276**] Let $*$ be a binary operation (not necessarily associative) on a set S satisfying

   - $x * (x * y) = y$ for all $x, y \in S$,
   - $(x * y) * y = x$ for all $x, y \in S$.

   Prove that $*$ is commutative. Find an example showing that $*$ need not be associative.

   Hint: Consider $(a * b) * ((a * b) * b)$ and simplify it two different ways. For an example, consider the operation on non-zero real numbers defined by $x * y := \frac{1}{xy}$.

8. Suppose that $f(x)$ is a polynomial with real coefficients. Prove that the range of $f(x)$ must be one of the following form:

   $$\{a\}; \quad [a, \infty); \quad (-\infty, a]; \quad (-\infty, \infty);$$

   where a is a real number.

   Hint: The Intermediate Value Theorem shows that the range must be connected.

9. [Putnam **1969 A1**] If $\mathbb{R}^2$ is the usual real plane with points (x, y), $-\infty < x, y < \infty$, and $p : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial with real coefficients, what are the possibilities for the image $p(\mathbb{R}^2)$?