LSU Problem Solving Seminar - Fall 2015
Dec. 2: Putnam Problems

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Putnam Mathematical Competition, **Sat., Dec. 5**, Lockett Hall 244, 8:30 A.M. – 5:00 P.M.

Test-taking tips:

- **Format.** The Exam is given in two 3-hour sessions of 6 problems each, with a lunch break from 12:00 – 2:00 P.M. The morning session's problems are labeled **A1 – A6**, and the afternoon's **B1 – B6**.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of 120. Typically there is very little partial credit given, and a submitted problem will receive **0, 1, 2, 9,** or **10** points.
- **A1/A2/B1.** In recent years these three problems have been the "easiest" part of the exam. More generally, the problems in each session are roughly ordered by difficulty. This is not an absolute rule, but you should expect that **A1** will have a relatively short solution, whereas **A6** will not. You should plan on spending at least 15 minutes each on **A1, A2, B1** before moving on to the rest of the Exam.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending approximately one hour writing your solution. Remember, it is better to solve one problem completely than several problems partially.

Warm Up:

1. (a) Three stacks of coins are placed on the table, consisting of $11, 7$, and 6 coins. You are allowed to move coins from one stack to another, but only if the destination stack is **exactly** doubled. For example, one possible initial move would be to move 6 coins from the first stack onto the third stack, but it is not allowed to move any coins from the second stack to the first (since this would require 11 coins from the second stack). Find a series of moves that results in three equal stacks, each containing 8 coins.

   (b) Is there a series of moves that results in two stacks of 12 coins?

2. (a) What was the last year that reads the same upside-down? When will the next such year be?

   (b) What was the last *palindromic* (reads the same forwards and backwards) year? When will the next one be?

3. There is a large class of logic puzzles known as "Knights and Knaves". In this setting, Knights always tell the truth, and Knaves always lie.

   (a) Alice says "Both Bob and I are Knaves". Determine whether each person is a Knight or a Knave.

(b) Alice says “Bob is a Knave”, and Bob says Neither Alice nor I are knaves. Determine whether each person is a Knight or a Knave.

(c) You arrive at a fork in the road where each of the two paths has a guard. You know that one guard is a Knight and one is a Knave, but you don’t know which is which. One road leads to Death, while the other leads to Freedom. By asking a question of just one guard, can you determine the road to Freedom?

Main Problems:

4. If A is a set of real numbers, its *sumset* is

$$A + A := \{a_1 + a_2 \mid a_1, a_2 \in A\}.$$

For example, if $A = \{1, 3, 5\}$, then $A + A = \{2, 4, 6, 8, 10\}$.

(a) If A has n elements, what is the largest possible number of elements in $A + A$?

(b) If A has n elements, what is the smallest possible number of elements in $A + A$?

Hint: Consider the largest element of A and use an inductive argument.

(c) Show that your bounds are sharp by giving specific examples of A.

Hint: What if $A = \{1, 2, \ldots, n\}$?

5. [Putnam **1992 B1**] Fix $n \geq 2$. For any set S of n real numbers, let A_S be the set of averages of two distinct elements of S. What is the minimum, over all S, of the size of A_S?

6. Find real differentiable functions f and g such that $(fg)' = f'g'$.

Hint: Try some simple examples – is this possible if f and g are both polynomials? What if they are exponentials, $f(x) = e^{ax}, g(x) = e^{bx}$? What if they are trigonometric functions?

7. [Putnam **1951 B2**] Find an example of real differentiable functions f and g such that $(f/g)' = f'/g'$.

8. (a) Suppose that $n \geq 1$. Prove that the number of positive integer pairs (i, j) such that $i \cdot j \leq n$ is at least

$$\frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n} > \ln(n) - 1.$$

(b) It is a fact from number theory (going back to Euclid) that there are infinitely many primes; note that if p is prime, then there are only two pairs (i, j) such that $i \cdot j = p$. Use part (a) to show that on average, a large integer n has $\ln(n) - 1$ pairs (i, j) such that $ij = n$.

9. [Putnam **1985 B3**] Let

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be a doubly infinite array of positive integers such that each positive integer appears exactly 8 times. Prove that $a_{mn} > mn$ for some integers m, n.