LSU Problem Solving Seminar - Fall 2015
Sep. 9: Calculus

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Useful facts:

- **Intermediate Value Theorem.** Suppose that $f(x)$ is a continuous function defined on the interval $[a, b]$, and r is a value in between $f(a)$ and $f(b)$, so that

$$f(a) < r < f(b) \quad \text{or} \quad f(a) > r > f(b).$$

  Then there is some point c in the interior of the interval, $a < c < b$, such that $f(c) = r$.

  In other words, a continuous function cannot "skip" any values.

- **Mean Value Theorem.** Suppose that $f(x)$ is differentiable on the interval $[a, b]$. Then there is a point $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  In other words, a differentiable function must achieve it's **average** *slope at some point.*

- **Critical Points.** If $f(x)$ is a differentiable function on an interval $[a, b]$, then its maxima/minima must occur at the end points or the **critical points**, where are those x such that $f'(x) = 0$.

Warm Up:

1. Suppose that a and b are positive real numbers such that $a + b = 100$. What is the maximum value of ab?

2. You are given a collection of ropes that each take exactly one hour to burn from end to end, although the speed of the flame is not necessarily the same throughout the rope.

   (a) How can you measure exactly 30 minutes using one rope?

   (b) How can you measure exactly 45 minutes using two ropes?

3. On Monday a cyclist rides along the Great River Road from Baton Rouge to New Orleans (100+ miles!), spending the night after his travels. On Tuesday he returns to Baton Rouge along the same route. Prove that there is some point along the Road that he passes at exactly the same time on both days.

   Hint: For an intuitive understanding, imagine superimposing his Monday trip and his Tuesday trip onto the same day. You should also try to write a formal proof using the Intermediate Value Theorem.

Main Problems:

4. A rancher wishes to pen in his herd in the fertile grass along a straight portion of the river. There are two permanent poles along the river that are exactly 100 feet apart; he has brought a length of 200 feet of rope and a single movable pole. If he ties the two ends of his rope to the permanent poles, and then creates a triangle by wrapping the rope around the movable pole, what is the maximal grazing area that can be enclosed?

5. A car travels 10 miles in 10 minutes.

   (a) Prove that at some point along the drive the speedometer read exactly 60 miles per hour.

   (b) Prove that somewhere along the route the car traveled 1 mile in exactly 1 minute.
   Hint: Let $f(x)$ denote the distance traveled after x minutes, and define $g(x) := f(x+1) - f(x)$. What can you say about $g(0), g(1), \ldots, g(9)$?

   (c) Show that it is possible that the car did **not** travel any 3 mile segment in exactly 3 minutes.

6. (a) Suppose that a and b are non-negative real numbers such that $a^2+b^2 = 100$. What is the minimum value of $a+b$?

   (b) Suppose that a and b are positive real numbers such that $ab = n$. What is the maximum value of a^b?

7. [Gelca-Andreescu **380**] Find the real parameters m and n such that the graph of the function $f(x) = \sqrt[3]{8x^3 + mx^2} - nx$ has the horizontal asymptote $y = 1$.

8. [Putnam **1956 A1**] Find
$$\lim_{x\to\infty} \left(\frac{a^x - 1}{ax - x}\right)^{\frac{1}{x}},$$
where $a \neq 1$ is a positive real number.