Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial with real coefficients. The degree of such a polynomial is the exponent of the leading term, in this case \( n \). A root of \( f \) is a value \( r \) such that \( f(r) = 0 \).

Useful Facts:

- **Rational Roots Test.** If all of the \( a_i \) are integers and \( r = \frac{p}{q} \) is a root, then \( p \) is a divisor of \( a_0 \) and \( q \) is a divisor of \( a_n \).
- **Fundamental Theorem of Algebra.** A polynomial of degree \( n \) has exactly \( n \) complex roots, counted with multiplicity. In particular, it has at most \( n \) real roots. Furthermore, if the roots are \( r_1, \ldots, r_n \), then \( f(x) = c(x - r_1) \cdots (x - r_n) \) for some constant \( c \).
- **Descartes' Rule of Signs.** If the non-zero coefficients of \( f(x) \) change sign \( s \) times, then \( f \) has at most \( s \) positive roots (with multiplicity). The actual number of positive roots is less than \( s \) by some multiple of 2. Replacing \( x \) by \( -x \) gives a similar test for negative roots.
- **Polynomial Division.** A polynomial \( f(x) \) is a multiple of \( g(x) \) if \( f(x) = h(x) \cdot g(x) \) for some polynomial \( h(x) \). If \( f(x) \) is not a multiple of \( g(x) \), then there are polynomials \( q(x) \) ("quotient") and \( r(x) \) ("remainder") such that \( f(x) = q(x) \cdot g(x) + r(x) \), where \( r(x) \) has lower degree than \( g(x) \).

---

**Warm Up:**

1. Factor the following polynomials:
   - (a) \( x^2 + 9x + 9 \);
   - (b) \( 6x^3 + x^2 - 5x - 2 \);
   - (c) \( x^3 - \frac{x^2}{2} + 3x - \frac{3}{2} \).

2. Let \( f(x) := x^3 + ax + 1 \), where \( a \) is some real number.
   - (a) Prove that \( f(x) \) always has at least one real root.
   - (b) Prove that if \( a \) is positive, then \( f(x) \) has exactly one real root.
     
     Optional: Determine the values of \( a \) such that \( f(x) \) has **more** than one real root.

3. Find all polynomials \( f(x) \) that satisfy \( f(x + 1) = f(x) + 2 \) for all \( x \).

---

**Main Problems:**

4. (a) Let \( f(x) = x^7 - 2x^6 - 2x^4 + 4x^3 + x \) and \( g(x) = x^2 - 3x + 2 \). Prove that \( f(x) \) is not a multiple of \( g(x) \), but that there is a constant \( c \) such that \( f(x) + c \) is a multiple of \( g(x) \). Find the value of \( c \).
(b) Determine whether or not \( f(x) = x^{2015} - x^{210} - x^{15} + x^5 + x^2 - x \) is a multiple of \( g(x) = x^3 - x \).

**Hint:** For both parts, do not try to divide \( f(x) \) by \( g(x) \) directly – the quotients are complicated! Instead, use the Fundamental Theorem of Algebra to find the remainders.

5. A root \( r \) of a polynomial \( f(x) \) is a **repeated root** of order \( k \) if \( f(r) = 0, f'(r) = 0, \ldots, f^{(k-1)}(r) = 0 \). Prove that if this is the case, then \( f(x) \) is a multiple of \( (x - r)^k \).

6. (a) **Equality Test.** Suppose that \( f(x) \) and \( g(x) \) are known to be polynomials of degree at most \( n \). Prove that if they agree on \( n + 1 \) different values, so that \( f(x_1) = g(x_1), \ldots, f(x_{n+1}) = g(x_{n+1}) \), then the polynomials are identical.

(b) Define a cubic polynomial by
\[
f(x) = \frac{(x - 1)(x - 2)(x - 3)}{(-1) \cdot (-2) \cdot (-3)} + \frac{x(x - 2)(x - 3)}{1 \cdot (-1) \cdot (-2)} + \frac{x(x - 1)(x - 3)}{2 \cdot 1 \cdot (-1)} + \frac{x(x - 1)(x - 2)}{3 \cdot 2 \cdot 1}.
\]
Show (by plugging in) that this polynomial satisfies \( f(0) = f(1) = f(2) = f(3) = 1 \). Then find the coefficients explicitly, determining the constants \( a, b, c, \) and \( d \) such that \( f(x) = ax^3 + bx^2 + cx + d \).

**Hint:** It is not necessary to do any messy calculations! Note that \( f(x) - 1 \) is a cubic polynomial with 4 different roots. . . .

7. [Gelca-Andreescu 148] Determine all polynomials \( P(x) \) with real coefficients for which there exists a positive integer \( n \) such that for all \( x \),
\[
P \left( x + \frac{1}{n} \right) + P \left( x - \frac{1}{n} \right) = 2P(x).
\]

**Hint:** What if \( P(x) \) is linear? What if it is quadratic?

8. [Putnam 1971 A2] Determine all polynomials \( f(x) \) that satisfy \( f(0) = 0 \) and \( f(x^2+1) = f(x)^2 + 1 \).