LSU Problem Solving Seminar - Fall 2015
Sep. 16: Polynomials

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Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be a polynomial with real coefficients. The *degree* of such a polynomial is the exponent of the leading term, in this case n. A *root* of f is a value r such that $f(r) = 0$.
Useful Facts:

- **Rational Roots Test.** If all of the a_i are integers and $r = \frac{p}{q}$ is a root, then p is a divisor of a_0 and q is a divisor of a_n.
- **Fundamental Theorem of Algebra.** A polynomial of degree n has exactly n complex roots, counted with multiplicity. In particular, it has at most n real roots. Furthermore, if the roots are $r_1, \ldots, r_n$, then $f(x) = c(x - r_1) \cdots (x - r_n)$ for some constant c.
- **Descartes' Rule of Signs.** If the non-zero coefficients of $f(x)$ change sign s times, then f has at most s positive roots (with multiplicity). The actual number of positive roots is less than s by some multiple of 2. Replacing x by $-x$ gives a similar test for negative roots.
- **Polynomial Division.** A polynomial $f(x)$ is a *multiple* of $g(x)$ if $f(x) = h(x) \cdot g(x)$ for some polynomial $h(x)$. If $f(x)$ is not a multiple of $g(x)$, then there are polynomials $q(x)$ ("quotient") and $r(x)$ ("remainder") such that $f(x) = q(x) \cdot g(x) + r(x)$, where $r(x)$ has lower degree than $g(x)$.

<u>Warm Up:</u>

1. Factor the following polynomials:
   (a) $x^2 + 9x + 9$;
   (b) $6x^3 + x^2 - 5x - 2$;
   (c) $x^3 - \dfrac{x^2}{2} + 3x - \dfrac{3}{2}$.
2. Let $f(x) := x^3 + ax + 1$, where a is some real number.
   (a) Prove that $f(x)$ always has at least one real root.
   (b) Prove that if a is positive, then $f(x)$ has exactly one real root.
   Optional: Determine the values of a such that $f(x)$ has **more** *than one real root.*
3. Find all polynomials $f(x)$ that satisfy $f(x + 1) = f(x) + 2$ for all x.

<u>Main Problems:</u>

4. (a) Let $f(x) = x^7 - 2x^6 - 2x^4 + 4x^3 + x$ and $g(x) = x^2 - 3x + 2$. Prove that $f(x)$ is not a multiple of $g(x)$, but that there is a constant c such that $f(x) + c$ **is** a multiple of $g(x)$. Find the value of c.

(b) Determine whether or not $f(x) = x^{2015} - x^{210} - x^{15} + x^5 + x^2 - x$ is a multiple of $g(x) = x^3 - x$.

Hint: For both parts, do not try to divide $f(x)$ by $g(x)$ directly – the quotients are complicated! Instead, use the Fundamental Theorem of Algebra to find the remainders.

5. A root r of a polynomial $f(x)$ is a *repeated root* of order k if $f(r) = 0, f'(r) = 0, \ldots, f^{(k-1)}(r) = 0$. Prove that if this is the case, then $f(x)$ is a multiple of $(x-r)^k$.

6. (a) [**Equality Test.**] Suppose that $f(x)$ and $g(x)$ are known to be polynomials of degree at most n. Prove that if they agree on $n+1$ different values, so that $f(x_1) = g(x_1), \ldots, f(x_{n+1}) = g(x_{n+1})$, then the polynomials are identical.

   (b) Define a cubic polynomial by

$$f(x) = \frac{(x-1)(x-2)(x-3)}{(-1)\cdot(-2)\cdot(-3)} + \frac{x(x-2)(x-3)}{1\cdot(-1)\cdot(-2)} + \frac{x(x-1)(x-3)}{2\cdot 1\cdot(-1)} + \frac{x(x-1)(x-2)}{3\cdot 2\cdot 1}.$$

   Show (by plugging in) that this polynomial satisfies $f(0) = f(1) = f(2) = f(3) = 1$. Then find the coefficients explicitly, determining the constants a, b, c, and d such that $f(x) = ax^3 + bx^2 + cx + d$.

   Hint: It is not necessary to do any messy calculations! Note that $f(x) - 1$ is a cubic polynomial with 4 different roots....

7. [Gelca-Andreescu **148**] Determine all polynomials $P(x)$ with real coefficients for which there exists a positive integer n such that for all x,

$$P\left(x + \frac{1}{n}\right) + P\left(x - \frac{1}{n}\right) = 2P(x).$$

Hint: What if $P(x)$ is linear? What if it is quadratic?

8. [Putnam **1971 A2**] Determine all polynomials $f(x)$ that satisfy $f(0) = 0$ and $f(x^2+1) = f(x)^2 + 1$.