LSU Problem Solving Seminar - Fall 2015
Sep. 23: Enumeration

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/2015-Putnam.html`

Useful facts: (n and k are non-negative integers)

- **Pigeonhole Principle.** If more than n objects are distributed amongst n sets, then some set contains multiple objects.
- **Permutations.** The number of ordered lists of k distinct elements chosen from a set of n objects is $P(n,k) := \frac{n!}{(n-k)!}$.
- **Binomial Coefficients.** Given two non-negative integers n and k, the number of ways of choosing k (unordered) objects from a set of n is $\binom{n}{k} := \dfrac{n!}{k!(n-k)!}$ (this is read as "n choose k"). They satisfy the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- **Binomial Theorem.** For an integer $n \geq 0$, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.
- **Number of subsets.** There are 2^n distinct subsets of a set with n elements.

Warm Up:

1. (a) The last four digits of one's Social Security Number is often used as a security measure for password recovery. Prove that at least **three** students at LSU have Social Security Numbers with the same last four digits.

   Note: There are more than 30,000 students at LSU, over 90% of whom are US citizens.

   (b) Approximately 135 million people are born each year worldwide. Prove that last year there were two babies born at the same time, down to the precise hour, minute, and second!

2. An ice-cream shop offers 6 flavors: Apple, Banana, Cherry, Dark Chocolate, Egg Yolk (Custard), and French Vanilla. How many distinct flavor combinations are there for each of the following?

   (a) A *Super Cone* consists of 3 scoops of distinct flavors. Since the top scoops will drip onto the bottom scoops, the order matters!

   (b) A *Sandwich Sampler* consists of 4 pairs of miniature cookies with the choice of ice cream in between. Each pair of cookies is different, so ice cream flavors can be repeated.

   (c) A *Blended Milkshake* consists of 2 distinct flavors mixed together.

   (d) A *Sundae* consists of any 3 scoops in a bowl, with an additional choice of 5 different toppings.

3. (a) How many ways are there to choose a Committee of **at most** 4 students from a class of 9 students?

(b) How many ways are there to choose a Committee of k students from a class of n?

(c) Now suppose that the Committee needs a President. If the k students are chosen first, and then the President is elected from within the Committee, then the total number of choices is $\binom{n}{k} \cdot k$. However, an alternative procedure would be to first elect the President from the entire class, and then choose the remaining $k-1$ Committee members – what is the formula in this case?

Remark: This is a "combinatorial/counting proof" of an identity. You should also try to prove it algebraically, using the formula for $\binom{n}{k}$.

<u>Main Problems:</u>

4. (a) Prove that

$$\binom{m}{k} + \binom{m-1}{k-1} + \cdots + \binom{m-k+1}{1} + \binom{m-k}{0} = \binom{m+1}{k}.$$

Remark: If you draw Pascal's Triangle, you can see why this is sometimes known as the "Hockey-Stick Identity"!

(b) [Gelca-Andreescu **880**] Prove that

$$1 \cdot \binom{n}{1}^2 + 2 \cdot \binom{n}{2}^2 + \cdots + n \cdot \binom{n}{n}^2 = n\binom{2n-1}{n-1}.$$

5. Prove that

$$(n-1) \cdot P(n,0) + (n-2) \cdot P(n,1) + \cdots + 1 \cdot P(n,n-2) = n! - 1.$$

Try to find several proofs: 1. Telescoping Sum, 2. Induction, 3. Combinatorial, ...

6. (a) Let $x_1, x_2, \ldots$ be a sequence of positive integers such that

$$x_1 = 1,\ x_2 = 2, \qquad \text{and } x_{j+1} - x_j = 1 \text{ or } 2 \text{ for } j \geq 3.$$

Prove that for any positive integer n, there are terms in the sequence such that $x_j - x_i = n$.

(b) [Gelca-Andreescu **35**] Let $x_1, x_2, \ldots$ be a sequence of positive integers such that

$$1 = x_1 < x_2 < x_3 < \cdots, \qquad \text{and } x_{j+1} \leq 2j \text{ for } j \geq 1.$$

Prove that for any positive integer n, there are terms in the sequence such that $x_j - x_i = n$.

Hint: Try to define n pigeonholes for the $x_1, x_2, \ldots, x_{n+1}$.

7. [Putnam **2000 B1**] Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume that for each j, at least one of a_j, b_j, c_j is odd. Show that there are integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4N}{7}$ values of j.

8. [Putnam **1993 A3**] Let $\mathcal{P}_n$ be the set of all subsets of $\{1, 2, \ldots, n\}$. Let $c(n,m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n.$$