- Virginia Tech Mathematics Contest. Sat., Oct. 24. **Sign-up deadline: Oct. 1**.
- Putnam Mathematical Competition. Sat., Dec. 5. **Sign-up deadline: Oct. 8**.

LSU Problem Solving Seminar - Fall 2015
Sep. 30: Integration

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/2015-Putnam.html`

Useful facts:

- **Partial Fractions.** If $f(x)$ is a polynomial whose degree is less than n, then there are constants $a_1, \ldots, a_n$ such that
$$\frac{f(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{x-r_1} + \cdots + \frac{a_n}{x-r_n}.$$
(Here the roots r_i must be distinct – there is a more complicated version for repeated roots.)
- **Fundamental Theorem(s) of Calculus.** Suppose that $f(x)$ is a continuous function.
  - If $F(x)$ is an antiderivative of f, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.
  - Define $g(x) := \displaystyle\int_a^x f(t)dt$. Then $g'(x) = f(x)$.
- **Integration By Parts.** Suppose that f and g are differentiable. Then
$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$
- **Symmetries and Substitution.** Remember, integration problems on Mathematics Contests are meant to have solutions! A complicated integral often has a hidden symmetry or substitution that makes it much easier to evaluate. For example, if $f(x)$ is an *odd* function, then $\int_{-a}^a f(x)\, dx = 0$.

Warm Up:

1. Calculate the following antiderivatives (indefinite integrals):
   (a) $\displaystyle\int \sin(2x)\, dx$,
   (b) $\displaystyle\int \frac{x}{x^2+1}\, dx$
   (c) $\displaystyle\int \ln(x)\, dx$,
   Hint: Try integration by parts.

2. (a) Evaluate $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$.

   (b) Evaluate $\displaystyle\int_{0}^{2} (x+1)\cos\left((x+1)^2\right)\,dx$.

   Hint: Both of these can be answered with minimal computation.

3. Let $\alpha := \displaystyle\int_{2}^{3} \frac{x^2+1}{x^3-x}\,dx$. There is a rational number $\frac{a}{b}$ such that $e^{\alpha} = \frac{a}{b}$. Find the values of a and b.

   Hint: Use a partial fractions decomposition.

Main Problems:

4. (a) Suppose that $f(x)$ is a function such that $f(x) \neq -f(-x)$ at any point. For $a > 0$, let

$$I := \int_{-a}^{a} \frac{f(x)}{(f(x)+f(-x))}\,dx.$$

   Find the value of I.

   Hint: Make the substitution $x = -u$ and compare to I.

   (b) [See VTRMC **2012 # 1**] Evaluate

$$\int_{0}^{\frac{\pi}{2}} \frac{\sin^2(x)+\sin(x)\cos(x)}{1+\sin(2x)}\,dx.$$

   Hint: See part (a).

5. Find the function $y(x)$ that satisfies

$$y' = -2xy^2, \quad y(0) = \frac{1}{4}.$$

   On what range is this solution valid?

6. (a) A continuous function $f(x)$ is *nonincreasing* if for any two points $x < y$, $f(x) \geq f(y)$. Prove that if $f(x)$ is differentiable, this condition is equivalent to $f'(x) \leq 0$ for all x.

   (b) Suppose that $f(x) \leq 0$ for all x, and define

$$g(x) := \int_{0}^{x} f(t)dt.$$

   Prove that $g(x)$ is nonincreasing.

   (c) [Gelca-Andreescu **410**] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For $x \in \mathbb{R}$ we define

$$g(x) := f(x)\int_{0}^{x} f(t)dt.$$

   Prove that if $g(x)$ is a nonincreasing function, then $f(x) = 0$ for all x.

7. [Putnam **1990 B1**] Find all real-valued continuously differentiable functions f on the real line such that for all x

$$\left(f(x)\right)^2 = \int_{0}^{x} \left(f(t)^2 + f'(t)^2\right)\,dt + 1990.$$