LSU Problem Solving Seminar - Fall 2015
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Useful facts:

- **Law of Cosines.** If a triangle has sides of lengths a, b, and c, and α is the angle opposite the side of length a, then
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
- **Law of Sines.** If β is the angle opposite b, and γ is the angle opposite c, then
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R},$$
where R is the radius of the circumscribed circle (which contains the vertices of the triangle).
- For all x,
$$\sin^2(x) + \cos^2(x) = 1,$$
- **Addition Formulas.** For all x and y,
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
- **Heron's Formula.** If a triangle has side lengths a, b, and c, then its area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s := \dfrac{a+b+c}{2}$ is the *semiperimeter.*

Warm Up:

1. (a) A cube with side-length 3 is to be cut into 27 smaller cubes of side-length 1. What is the minimum number of cuts required? You are allowed to move/stack pieces in between each cut.
   (b) A cube with side-length 5 is painted Red on all of its faces. It is then cut into smaller cubes of side-length 1. How many of the smaller cubes have exactly 2 Red faces? How many have 1 Red face?

2. (a) Prove that the sum of the angles in a polygon with n vertices is $(n-2)\cdot 180°$.
   (b) A *regular n-gon* is a polygon with n vertices such that all sides are equal (equivalently, all of the central angles are also equal). Consider the regular n-gon inscribed in the unit circle, so that the distance from the center to each vertex is 1. Let A_n denote the area of the n-gon. Prove that
$$\frac{A_{2n}}{A_n} = \sec\left(\frac{\pi}{n}\right).$$
   Hint: It is possible to solve this problem without writing down an explicit formula for A_n.

3. Find constants α, β such that

$$\sin(x) + \sqrt{3}\cos(x) = \alpha \sin(x + \beta).$$

<u>Main Problems:</u>

4. In a triangle ABC, suppose that the measure of angle C is twice the measure of angle A. If the length of $\overline{AB}$ is 6, and the length of $\overline{CB}$ is 4, determine the length of $\overline{AC}$.

5. [VTRMC **2001 # 2**] Two circles of radius 1 and 2 are placed so that they are tangent to each other and a line. A third circle is nestled in between them so that it is tangent to the two circles and line. Find the radius of the third circle.

6. (a) Let $ABCD$ be a quadrilateral. Prove that it is a parallelogram if and only if the midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$ are identical.

   (b) [From Gelca-Andreescu **589**] Let $ABCD$ be a quadrilateral, and let M, N, P, and Q be the midpoints of the four sides. Determine conditions on $ABCD$ such that $MNPQ$ is a parallelogram.

7. Suppose that every point in the plane is colored Red, White, or Blue. Prove that there are two points of the same color at a distance of exactly 1 unit apart.

   Hint: Consider the vertices of equilateral triangles with side-length 1*.*

8. [Putnam **1957 A5**] Let S be a set of n points in the plane such that the distance between any two points is at most 1. Show that there are at most n pairs of points in S that are at a distance of exactly 1.