Useful facts:

- **Law of Cosines.** If a triangle has sides of lengths $a, b,$ and $c,$ and $\alpha$ is the angle opposite the side of length $a,$ then
  \[ a^2 = b^2 + c^2 - 2bc \cos(\alpha). \]

- **Law of Sines.** If $\beta$ is the angle opposite $b,$ and $\gamma$ is the angle opposite $c,$ then
  \[ \frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R}, \]
  where $R$ is the radius of the circumscribed circle (which contains the vertices of the triangle).

- For all $x,$
  \[ \sin^2(x) + \cos^2(x) = 1, \]

- **Addition Formulas.** For all $x$ and $y,$
  \[ \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y), \]
  \[ \sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x). \]

- **Heron’s Formula.** If a triangle has side lengths $a, b,$ and $c,$ then its area is
  \[ A = \sqrt{s(s - a)(s - b)(s - c)}, \]
  where $s := \frac{a + b + c}{2}$ is the semiperimeter.

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Warm Up:

1. (a) A cube with side-length 3 is to be cut into 27 smaller cubes of side-length 1. What is the minimum number of cuts required? You are allowed to move/stack pieces in between each cut.

   (b) A cube with side-length 5 is painted Red on all of its faces. It is then cut into smaller cubes of side-length 1. How many of the smaller cubes have exactly 2 Red faces? How many have 1 Red face?

2. (a) Prove that the sum of the angles in a polygon with $n$ vertices is $(n - 2) \cdot 180^\circ$.

   (b) A regular $n$-gon is a polygon with $n$ vertices such that all sides are equal (equivalently, all of the central angles are also equal). Consider the regular $n$-gon inscribed in the unit circle, so that the distance from the center to each vertex is 1. Let $A_n$ denote the area of the $n$-gon. Prove that
   \[ \frac{A_{2n}}{A_n} = \sec \left( \frac{\pi}{n} \right). \]

   *Hint: It is possible to solve this problem without writing down an explicit formula for $A_n$. *
3. Find constants $\alpha, \beta$ such that

$$\sin(x) + \sqrt{3}\cos(x) = \alpha \sin(x + \beta).$$

Main Problems:

4. In a triangle $ABC$, suppose that the measure of angle $C$ is twice the measure of angle $A$. If the length of $AB$ is 6, and the length of $CB$ is 4, determine the length of $AC$.

5. [VTRMC 2001 # 2] Two circles of radius 1 and 2 are placed so that they are tangent to each other and a line. A third circle is nestled in between them so that it is tangent to the two circles and line. Find the radius of the third circle.

6. (a) Let $ABCD$ be a quadrilateral. Prove that it is a parallelogram if and only if the midpoints of the diagonals $AC$ and $BD$ are identical.

(b) [From Gelca-Andreescu 589] Let $ABCD$ be a quadrilateral, and let $M, N, P,$ and $Q$ be the midpoints of the four sides. Determine conditions on $ABCD$ such that $MNPQ$ is a parallelogram.

7. Suppose that every point in the plane is colored Red, White, or Blue. Prove that there are two points of the same color at a distance of exactly 1 unit apart.

*Hint: Consider the vertices of equilateral triangles with side-length 1.*

8. [Putnam 1957 A5] Let $S$ be a set of $n$ points in the plane such that the distance between any two points is at most 1. Show that there are at most $n$ pairs of points in $S$ that are at a distance of exactly 1.