- Virginia Tech Mathematics Contest. Sat., Oct. 24: **Coates 103, 8:30 – 11:30 A.M.**
- Putnam Mathematical Competition. Sat., Dec. 5.

LSU Problem Solving Seminar - Fall 2015
Oct. 14: Inequalities

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Website: `www.math.lsu.edu/~mahlburg/teaching/2015-Putnam.html`

Useful facts:

- **Arithmetic-Geometric Mean Inequality.** If $a_1, \ldots, a_n$ are non-negative real numbers, then
$$\sqrt[n]{a_1 \cdots a_n} \le \frac{a_1 + \cdots + a_n}{n}.$$
Furthermore, the right side is strictly larger than the left unless all of the a_i are equal.
- **Hölder's p-norm Inequality.** If $0 < p < q$ and $a_1, \ldots, a_n$ are non-negative real numbers, then
$$\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \le \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}},$$
with strict inequality unless all of the a_i are equal.
- **Cauchy-Schwarz Inequality.** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then
$$\left(a_1 b_1 + \cdots + a_n b_n\right)^2 \le \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$
Furthermore, the right side is strictly larger unless $(b_1, \ldots, b_n) = (\lambda a_1, \ldots, \lambda a_n)$ for some real λ. Written in vector notation and Euclidean distance, $|\overrightarrow{a} \cdot \overrightarrow{b}|^2 \le |\overrightarrow{a}|^2 \cdot |\overrightarrow{b}|^2$.
- **Triangle Inequality.** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then
$$\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \le \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$$
Written in vector notation, $|\overrightarrow{a} + \overrightarrow{b}| \le |\overrightarrow{a}| + |\overrightarrow{b}|$.

Warm Up:

1. For each of the following pairs, determine which expression is larger (without using a calculator!):

   (a) $5^{4^{3^{2^1}}}$ or $1^{2^{3^{4^5}}}$? What about: $5^{4^{3^2}}$ or $2^{3^{4^5}}$?

   (b) 2015^{2014} or 2014^{2015}?

   (c) $\sqrt{2000} + \sqrt{15}$ or $\sqrt{2001} + \sqrt{14}$?

   (d) $e^{1 + \frac{1}{2} + \cdots + \frac{1}{2015}}$ or 2015?

   You may use a calculator to check data for this last one!

2. (a) Find the minimum value of $9x + 1 + \dfrac{4}{x}$ for positive x.

   (b) If a, b, c are positive real numbers, show that
$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

3. In this problem you will learn a shortcut for calculating/estimating cumulative averages, which can be helpful for your course grades!

   (a) A student has an average score of 82 on 4 previous homework assignments. If she scores 92 on the fifth assignment, what is her new average?

   (b) Notice that in part (a), the new assignment was 10 points higher than the previous average, but this effect must be spread over five assignments. Thus the "scaled difference" is $\frac{10}{5} = 2$. How does this compare to the old and new averages from part (a)?

   (c) Prove the general case: If $\overline{x}$ is the average of $x_1, \ldots, x_{n-1}$, and $x_n - \overline{x} = \Delta$, then the average of $x_1, \ldots x_n$ is $\overline{x} + \dfrac{\Delta}{n}$.

Main Problems:

4. Use the Arithmetic-Geometric Mean Inequality **instead** of calculus:

   (a) Find the minimum value over all positive x of the expression
$$\frac{x^2}{4} + \frac{x}{2} - 2 + \frac{8}{x^3}.$$

   (b) A wooden box with volume 250 cubic feet is to be built, and for support the bottom side must be three times thicker than the others. Find the dimensions that minimize the amount of wood used. How many square feet of wood are needed?

5. Suppose that $0 \leq x_i \leq 1$ for $i = 1, \ldots, n$.

   (a) Prove that
$$(1+x_1)\cdots(1+x_n) \leq (1+\overline{x})^n,$$
   where $\overline{x}$ is the average of the x_i.

   (b) Prove that
$$(1+x_1)\cdots(1+x_n) \leq 2^{n-1}(1+x_1x_2\cdots x_n),$$

6. [VTRMC **1981 # 6**] With k a positive integer, prove that $\left(1 - \dfrac{1}{k^2}\right)^k \geq 1 - \dfrac{1}{k}$.

7. Suppose that a collection of n discs cover a line segment of length L. Let A be the sum of the areas of the discs. Prove that $nA \geq \dfrac{\pi L^2}{4}$.

   Hint: Cauchy-Schwarz.

8. [Putnam **1950 B1**] Let $P_1, \ldots, P_n$ be points on a line, not necessarily distinct. Which points P minimize the sum of the distances $|PP_i|$?