Virginia Tech Regional Math Contest: Saturday, Oct. 24. Location: Coates Hall 103, LSU. You **must** be present promptly at 8:30 A.M., or you will not be able to enter the building.

LSU Problem Solving Seminar - Fall 2015
Oct. 21: Number Theory

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Website: `www.math.lsu.edu/~mahlburg/teaching/2015-Putnam.html`

Useful facts:

- The prime factorization of the current year is $2015 = 5 \cdot 13 \cdot 31$, and the previous year was $2014 = 2 \cdot 19 \cdot 53$.
- **Fermat's Little Theorem.** If p is prime and a is any integer, then $a^p - a$ is a multiple of p.
- For any integer n, its square can only have the following remainders:

  0 or 1 when divided by 3; 0 or 1 when divided by 4; 0, 1, 4, 5, 6, or 9 when divided by 10.

- **Greatest Common Divisors.** If a and b are integers with prime factorizations $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $b = p_1^{\beta_1} \cdots p_r^{\beta_r}$, their greatest common divisor is

$$\gcd(a,b) = p_1^{\min\{\alpha_1,\beta_1\}} \cdots p_r^{\min\{\alpha_r,\beta_r\}}.$$

  The equation $ax + by = N$ has integer solutions if and only if $\gcd(a,b)$ divides N.

- **Euler's Totient Function.** If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then define

$$\phi(n) := n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

  Among the integers $1, 2, \ldots, n$, exactly $\phi(n)$ of them satisfy $\gcd(a,n) = 1$. Furthermore, if $\gcd(a,n) = 1$, then $a^{\phi(n)}$ has remainder 1 when divided by n.

- **Casting Out Nines.** If n is an integer and s is the sum of its decimal digits, then $n - s$ is a multiple of 9.

Warm Up:

1. (a) Show that $n^3 - n$ is a multiple of 6.

   (b) Show that $n^7 - n$ is a multiple of 42.

2. A long hallway contains Rooms numbered by $1, 2, \ldots, 100$, all of which are initially closed. A line of people then walks down the hallway one-by-one. The first person opens every door; the second person then closes every second door; the third person stops at every third door and reverses its status, opening it if it was closed, and closing it if open. This continues until a hundredth person, who reverses the position of the 100-th door. Which doors are open at the end of this process?

   Hint: First try the procedure on a smaller number of doors.

3. (a) Determine the remainder when 2013^{2015} is divided by 2014.
   (b) Determine the remainder when 2015^{2013} is divided by 2014.

<u>Main Problems:</u>

4. (a) Let n be an integer with remainder 3 when divided by 4, so that $n = 4m + 3$ for some n. Show that n is divisible by a prime of the form $p = 4k + 3$.
   (b) Prove that there are infinitely many primes of the form $p = 4k + 3$.
   Hint: If there were finitely many $p_1, \ldots, p_r$, consider $4p_1 \cdots p_r + 3$.

5. If $n \geq 1$, prove that the product $2n \cdot (2n-1) \cdots (n+2) \cdot (n+1)$ is *exactly* divisible by 2^n. This means that

$$2n \cdot (2n-1) \cdots (n+2) \cdot (n+1) = 2^n \cdot m$$

for some odd integer m.

6. (a) Prove that $2^9 + 5^9$ is a multiple of 7.
   (b) Prove that $15^{15} + 45^{45}$ is a multiple of 2015.
   Hint: Use the prime factorization of 2015.

7. A polynomial $p(x)$ is *integer-valued* if $p(n)$ is an integer for any integer n. Throughout this problem, let a, b, c be distinct integers.
   (a) Find an integer-valued polynomial such that $p(a) = b$ and $p(b) = a$. Characterize all such polynomials.
   (b) Prove that if $p(x)$ has integral coefficients, then $p(b) - p(a)$ is a multiple of $b - a$.
   (c) Prove that there is no polynomial $p(x)$ with integral coefficients such that $p(a) = b, p(b) = c$ and $p(c) = a$.
   Remark: However, there can be integer-valued polynomials with this property. For example, $p(x) = -\frac{3}{2}x^2 + \frac{11x}{2} - 2$ satisfies $p(1) = 2, p(2) = 3, p(3) = 1$, and is integer-valued!

8. [VTRMC **2009 # 6**] Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square.

9. (a) Show that there are no integer solutions to $x^2 + 5y^3 = 202$.
   (b) Show that there are no integer solutions to $x^2 + 4xy + 11y^2 = 80$.
   Hint: Complete the square.

10. [Putnam **1954 A7**] Prove that the equation $m^2 + 3mn - 2n^2 = 122$ has no integral solutions.