LSU Problem Solving Seminar - Fall 2016
Oct. 26: Virginia Tech Math Contest Review

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/2016-Putnam.html`

This week's practice sheet provides a detailed look at several of the problems from last weekend's 2016 Virginia Tech Regional Math Contest. Each contest problem is **preceded** by a related problem that illustrates some relevant techniques in a somewhat easier context.

1. Recall the following trigonometric identities and derivatives:

$$\tan^2(u) + 1 = \sec^2(u),$$
$$\frac{d}{du}\tan(u) = \sec^2(u), \qquad \frac{d}{du}\sec(u) = \sec(u)\tan(u).$$

   (a) Use the substitution $x = \tan(u)$ to evaluate the integral $\displaystyle\int_0^{\sqrt{3}} \frac{1}{x^2+1} dx$.

   (b) Calculate the antiderivative $\displaystyle\int \tan(x) dx$ by first writing $\tan(x) = \frac{sec(x)\tan(x)}{\sec(x)}$, and then making the substitution $v = \sec(x)$.

   (c) Calculate $\frac{d}{dx} \ln|\cos(x)|$. Compare your answer to 1(b). Why are the absolute value signs necessary? For what values of x is this derivative defined?

2. [VTRMC **2016 # 1**] Evaluate $\displaystyle\int_1^2 \frac{\ln x}{2 - 2x + x^2}\, dx$.

   Hint: The integrand does not have a closed-form antiderivative, so you will need to identify cancellations or symmetries. Use a trigonometric substitution, and recall the sine addition formula: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

3. (a) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges if and only if $k > 1$.

   (b) **Limit Comparison Test.** Suppose that a_n and b_n are non-negative series that are *asymptotically equal*, which we write as $a_n \sim b_n$ as $n \to \infty$. This means precisely that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

   (c) The Taylor expansion of the exponential function is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

   Plug in n and compare appropriate terms to conclude that $n! > \left(\frac{n}{e}\right)^n$. In fact, Stirling's approximation states that for large n, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

4. [VTRMC **2016 # 2**] Determine the real numbers k such that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(2n)!}{4^n n! n!}\right)^k$ is convergent.

5. Fermat's Little Theorem states that if a is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$ (which means that $a^{p-1} - 1$ is a multiple of p).

(a) Suppose that p is an odd prime and a^2+1 is a multiple of p, so $a^2 \equiv -1 \pmod{p}$. Then

$$a^{p-1} = \left(a^2\right)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \pmod{p}.$$

Now show that if p is a prime of the form $p = 4k+3$, then $1 \equiv -1 \pmod{p}$, which is impossible. Thus for these primes there are no such a.

Remark: It is a deeper fact that if $p = 4k+1$, then there is **always** *an a such that $a^2 \equiv -1 \pmod{p}$.*

(b) Show that if a^2+1 is a multiple of p, then $(a-p)^2+1$ is also a multiple of p.

(c) Conclude that if a^2+1 is a multiple of p, then there is some value a' such that the largest prime factor of a'^2+1 is p.

Hint: Find an a' such that $a' < p$ and a'^2+1 is a multiple of p.

6. [VTRMC **2016 # 4**] For a positive integer a, let $P(a)$ denote the largest prime divisor of a^2+1. Prove that there exist infinitely many triples (a,b,c) of distinct positive integers such that $P(a) = P(b) = P(c)$.

7. Let $\alpha := 1+\sqrt{2}$, and for $n \geq 1$, define integers r_n and s_n by expanding $\alpha^n = r_n + s_n\sqrt{2}$. For example, $r_1 = 1$, $s_1 = 1$; since $\alpha^2 = 3+2\sqrt{2}$, the next values are $r_2 = 3$ and $s_2 = 2$.

   (a) Calculate the next several values of r_n, s_n.

   (b) Let $\overline{\alpha} := 1-\sqrt{2}$ (this is known as the *conjugate* of α). Show that $\overline{\alpha}^n = r_n - s_n\sqrt{2}$.

   (c) Use $\overline{\alpha}$ to show that $r_n^2 - 2s_n^2 = (-1)^n$ for all n.

   (d) By definition, $\alpha^{2n} = r_{2n} + s_{2n}\sqrt{2}$. However, it is also true that

   $$\alpha^{2n} = \alpha^n \cdot \alpha_n = \left(r_n + s_n\sqrt{2}\right)^2.$$

   Use this to obtain an expression for r_{2n} in terms of r_n and s_n.

   (e) Combine the previous two parts and conclude that $r_{2n} - (-1)^n$ is always a perfect square!

8. [VTRMC **2016 # 5**] Suppose that m, n, r are positive integers such that

$$1 + m + n\sqrt{3} = (2+\sqrt{3})^{2r-1}.$$

Prove that m is a perfect square.

Remark: For any non-square integer N, there is a corresponding Pell's equation $x^2 - Ny^2 = 1$. It is a fact from Number Theory that all solutions arise as powers of a fundamental solution $(r+s\sqrt{N})(r-s\sqrt{N}) = 1$.

9. Recall the Binomial Theorem, which states that for $n \geq 0$, $(1+x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k$. Here the binomial coefficients are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

In this problem you will explore several proofs of Vandermonde's summation formula

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j},$$

which holds for all non-negative integers m, n, and k.

(a) *Series.* Note that $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Apply the Binomial Theorem to each product, and find the coefficient of x^k.

(b) *Combinatorial.* Suppose that a committee of k people is to be chosen from a group of m women and n men. One procedure would be to have all $m+n$ people in the same room and vote for the k representatives. A second procedure would be to first decide how many women the committee will have; call this number j. Then the women meet separately to choose j representatives, and the men choose $k-j$. Compare the expressions that arise from these two procedures to obtain the identity.

(c) *Calculus.* The iterated product rule states that

$$\frac{d^k}{dx^k}\left(f(x)\cdot g(x)\right) = \sum_{j=0}^{k} \binom{k}{j}\left(\frac{d^j}{dx^j} f(x)\right)\cdot\left(\frac{d^{k-j}}{dx^{k-j}} g(x)\right).$$

Apply $\dfrac{d^k}{dx^k}$ to both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$ and set $x = 0$.

(d) Show that $\displaystyle\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$

(e) What is the constant coefficient (i.e., x^0) of $(2+x+x^{-1})^n$?
Hint: $(1+x)(1+x^{-1}) =$?

10. [VTRMC **2016 # 7**] Let q be a real number with $|q| \neq 1$ and let k be a positive integer. Define a Laurent polynomial $f_k(X)$ in the variable X, depending on q and k, by $f_k(X) = \displaystyle\prod_{i=0}^{k-1}\left(1-q^iX\right)\left(1-q^{i+1}X^{-1}\right)$. Show that the constant term of $f_k(X)$, i.e. the coefficient of X^0 in $f_k(X)$, is equal to

$$\frac{\left(1-q^{k+1}\right)\left(1-q^{k+2}\right)\cdots\left(1-q^{2k}\right)}{(1-q)(1-q^2)\cdots(1-q^k)}.$$