LSU Problem Solving Seminar - Fall 2016
Nov. 2: Sequences and Series

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- **Geometric Series.** If $|x| < 1$, then
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$
- **Ratio Test.** Let $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $L < 1$, then $\sum_{n\geq 1} a_n$ converges, and if $L > 1$, then the sum diverges. If $L = 1$, the test is inconclusive.
- **Monotone Convergence.** If $a_1 \leq a_2 \leq \ldots$ and all $a_n \leq B$ for some constant B, then $\lim_{n\to\infty} a_n$ exists.
- **Alternating Series.** If $a_1 \geq a_2 \geq \ldots$ and $\lim_{n\to\infty} a_n = 0$, then $a_1 - a_2 + a_3 - \ldots$ converges.
- **Linear Recurrences.** The *characteristic polynomial* associated to a (homogeneous) recurrence $a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n$ is $p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0$. If $p(x)$ has **distinct** roots $\lambda_1, \ldots, \lambda_k$, then the general solution to the recurrence is
$$a_n = b_1\lambda_1^n + \cdots + b_k\lambda_k^n,$$
where the constants are determined by k initial values. If there is a **repeated** root λ of order m, then the general solution has the term $(d_{k-1}n^{k-1} + \cdots + d_1 n + d_0)\lambda^n$.

Warm Up

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1. Evaluate the following series:
   (a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = ?$
   (b) $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = ?$
   (c) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = ?$
   Hint: Apply partial fractions and write $\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$ *for constants* a *and* b.
2. Solve the following recurrences; this means finding a closed-form expression for a_n.
   (a) $a_1 = 1$ and $a_{n+1} = a_n + 2n + 1$ for $n \geq 1$.
   (b) $a_1 = 1, a_2 = 5$, and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 1$.

Main Problems

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3. Let $b_1 = 1, c_1 = 1$, and for $n \geq 1$ define two sequences by
$$b_{n+1} := b_n + 2c_n, \qquad c_{n+1} := b_n + c_n.$$

(a) Determine the limiting value of their ratio, $\lim_{n\to\infty} \dfrac{b_n}{c_n}$.

Hint: Let $a_n := \frac{b_n}{c_n}$, *and find a relationship between* a_{n+1} *and* a_n.

(b) How is the limit affected by a different starting value? What if $\frac{b_1}{c_1}$ is an arbitrary positive fraction?

4. Evaluate the series $\dfrac{1}{2} - \dfrac{1}{6} + \dfrac{1}{12} - \dfrac{1}{20} + \cdots$.

   Hint: Recall the Taylor series of the natural logarithm, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

5. A sequence is defined by $a_1 = 0, a_2 = 2$, and $a_{n+2} = 2a_{n+1} - a_n + 2n + 2$ for $n \geq 1$.

   Hint: Let $b_n := a_n - a_{n-1}$, *and find a recurrence for* b_n*; this should result in a simpler "polynomial part" than* $2n+2$. *Then similarly set* $c_n := b_n - b_{n-1}$.

6. The *Fibonacci numbers* are defined by $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n, \qquad \text{for } n \geq 1.$$

   (a) Prove that for $n \geq 1$,

$$F_n^2 + F_{n+1}^2 = F_{2n+1}^2.$$

   (b) Determine the limiting value of the ratio of successive Fibonacci numbers,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = ?$$

   Hint: Let $u_n := \frac{F_{n+1}}{F_n}$, *and find a relationship between* u_{n+1} *and* u_n.

7. (a) Prove that for all $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \big(1 + 2 + \cdots + n\big)^2.$$

   (b) Suppose that $a_1, a_2, \ldots$ are positive real numbers such that for all $n \geq 1$,

$$a_1^3 + a_2^3 + \cdots + a_n^3 = \big(a_1 + a_2 + \cdots + a_n\big)^2.$$

   Is the only solution the one given above? In other words, must it be true that $a_i = i$ for all i?

8. [VTRMC **1992 # 4**] Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $t_1 = 1$ and $t_{n+1} := \sqrt{1 + t_n}$ for $n \geq 1$. Show that t_n is increasing in n and find $\lim_{n\to\infty} t_n$.

   Remark: The limit of this sequence is sometimes written in the more striking form $\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$.

9. [Putnam **1973 A2**] Suppose that $a_n = \pm\dfrac{1}{n}$ for all $n > 0$, and $a_{n+8} > 0$ if and only if $a_n > 0$. Show that if exactly four of $a_1, a_2, \ldots, a_8$ are positive, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Is the converse true?