LSU Problem Solving Seminar - Fall 2016
Nov. 9: Inequalities

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- **Arithmetic-Geometric Mean Inequality.** If $a_1, \ldots, a_n$ are non-negative real numbers, then
$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$
Furthermore, the right side is strictly larger than the left unless all of the a_i are equal.
- **Hölder's p-norm Inequality.** If $0 < p < q$ and $a_1, \ldots, a_n$ are non-negative real numbers, then
$$\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}},$$
with strict inequality unless all of the a_i are equal.
- **Cauchy-Schwarz Inequality.** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then
$$\left(a_1 b_1 + \cdots + a_n b_n\right)^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$
Furthermore, the right side is strictly larger unless $(b_1, \ldots, b_n) = (\lambda a_1, \ldots, \lambda a_n)$ for some real λ. Written in vector notation and Euclidean distance, $|\vec{a} \cdot \vec{b}|^2 \leq |\vec{a}|^2 \cdot |\vec{b}|^2$.
- **Triangle Inequality.** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then
$$\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$$
Written in vector notation, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Warm Up

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1. For each of the following pairs, determine which expression is larger (without using a calculator!):
   (a) $6^{5^{4^{3^2}}}$ or $2^{3^{4^{5^6}}}$? What about: $3^{3^{3^3}}$ or 4^{4^4}?
   (b) 2016^{2015} or 2015^{2016}?
   (c) $\sqrt{16} - \sqrt{15}$ or $\sqrt{2016} - \sqrt{2015}$?
   (d) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2016}$ or 10?
2. (a) Find the minimum value of $12x + 1 + \frac{1}{3x}$ for positive x.
   (b) If a, b, c are positive real numbers, show that
$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9.$$

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3. Suppose that $k \geq 0$. Use the Cauchy-Schwarz inequality to find the maximum value of
$$f(x) = \frac{(x+k)^2}{x^2+1}.$$

4. Use the Arithmetic-Geometric Mean Inequality **instead** of calculus for the following problems:
   (a) Find the minimum value of $f(x) = x^2 + 2x - 3 + \dfrac{32}{x^3}$ over all positive x.
   (b) A closed box with volume 1000 cubic inches is to be built. Find the dimensions that minimize the amount of wood used. How many square inches of wood are needed?
   (c) Now a closed box with volume 3000 cubic inches is to be built, where the top and bottom are triply reinforced (this means that three layers of wood are used for bottom and top panels). What are the dimensions that minimize the amount of wood required?

5. Recall that the factorial of a natural number n is defined by $n! := n \cdot (n-1) \cdots 2 \cdot 1$. Prove that for $n > 1$,
$$n! < \left(\frac{n+1}{2}\right)^n.$$
Remark: You might recall from a previous sheet that $n! > \left(\frac{n}{e}\right)^n$.

6. (a) Prove the Cauchy-Schwarz inequality in the case $n = 2$:
$$\left(a^2+b^2\right)\left(c^2+d^2\right) \geq (ac+bd)^2.$$
   (b) Suppose that $a_1 + a_2 + \cdots + a_n = 1$. Prove that
$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}.$$

7. For $a, b, c > 0$, prove that
$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

8. [**Gelca-Andreescu 113**] Let a, b, c be the side lengths of a triangle with the property that for any positive integer n, the numbers a^n, b^n, c^n can also be the side lengths of a triangle. Prove that the triangle must be isosceles.

9. [Putnam **2003 A2**] Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that
$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} + (b_1 b_2 \cdots b_n)^{\frac{1}{n}} \leq \left[(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\right]^{\frac{1}{n}}.$$