Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 23 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Nov. 30.
- Putnam Mathematical Competition, **Sat., Dec. 3.** The Exam will take place in Lockett 244 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2016
Nov. 16: Polynomials and Complex Numbers

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Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be a polynomial with real coefficients. It is *monic* if the leading coefficient $a_n = 1$. The *degree* of a polynomial is the exponent of the leading term, in this case n. A *root* of f is a value r such that $f(r) = 0$.

- **Rational Roots Test.** If all of the a_i are integers and $r = \frac{p}{q}$ is a root, then p is a divisor of a_0 and q is a divisor of a_n.
- **Descartes' Rule of Signs.** If the non-zero coefficients of $f(x)$ change sign s times, then f has at most s positive roots (with multiplicity). The actual number of positive roots is less than s by some multiple of 2. Replacing x by $-x$ gives a similar test for negative roots.
- **Polynomial Division Algorithm.** A polynomial $f(x)$ is a *multiple* of $g(x)$ if $f(x) = h(x) \cdot g(x)$ for some polynomial $h(x)$. If $f(x)$ is not a multiple of $g(x)$, then there are polynomials $q(x)$ ("quotient") and $r(x)$ ("remainder") such that $f(x) = q(x) \cdot g(x) + r(x)$, where $r(x)$ has lower degree than $g(x)$.
- **Fundamental Theorem of Algebra.** A polynomial of degree n has exactly n complex roots, counted with multiplicity. In particular, it has at most n real roots. Furthermore, if the roots are $r_1, \ldots, r_n$, then $f(x) = c(x - r_1)\cdots(x - r_n)$ for some constant c.
- **Sum and Product of Roots.** If a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has roots (with repetition) $r_1, \ldots, r_n$, then
$$a_{n-1} = -(r_1 + \cdots + r_n); \qquad a_0 = (-1)^n r_1 \cdots r_n.$$
- **Roots of Unity.** The roots of $x^n - 1$ are $1, e^{\frac{2\pi i}{n}}, e^{\frac{2\cdot 2\pi i}{n}}, \ldots, e^{\frac{(n-1)\cdot 2\pi i}{n}}$. These can also be written as $1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}$, where $\zeta_n := e^{\frac{2\pi i}{n}}$. The previous property implies that
$$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = 0.$$
- **Euler's Formula.** For real x, $e^{ix} = \cos(x) + i\sin(x)$.

Warm Up

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1. Find the *rational factorization* of the following polynomials (this means all factors have rational coefficients):

   (a) $x^3 + 6x^2 + 12x + 8$;

(b) $2x^3 - 3x^2 + x - 2$;

(c) $2x^3 - x^2 - 5x + 3$.

2. (a) The polynomial $x^4 - 6x^3 + 7x^2 + 6x - 8$ has roots $-1, 2$, and 4. Without doing any polynomial division, find the fourth root.

   (b) Let $f(x) = x^4 - 21x + 8$ have roots r_1, r_2, r_3, r_4. Given that $r_1 + r_2 = 3$, find the factorization of f.

3. Find the rational factorization of $x^4 + 4$.

Main Problems

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4. Prove that
$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right) = -\frac{1}{2}.$$
It is **not** necessary to use trigonometric identities. Instead, consider $\zeta_7 + \zeta_7^2 + \cdots + \zeta_7^6$ and apply Euler's Formula.

5. One of the following polynomials is a multiple of $g(x) = x^3 - x$; determine which one:
$$x^{2016} - x^{216} + x^{21} + x^{16} - x^2 - x \qquad \text{or} \qquad x^{2016} - x^{216} + x^{21} + x^{16} - x^2 + x?$$
Hint: Do **not** *try to divide by $g(x)$ explicitly! Instead, use the Division Algorithm and plug in roots of g.*

6. (a) Find all real numbers x such that $(x-1)^3 = -x^3$.

   (b) Find all complex numbers z such that $(z-1)^3 = -z^3$.

7. [Putnam **1977 B1**] Evaluate the infinite product
$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

8. (a) Find a polynomial with integer coefficients that has the zero $\sqrt{2} + \sqrt{3}$.

   (b) [**Gelca-Andreescu 149**] Find a polynomial with integer coefficients that has the zero $\sqrt{2} + \sqrt[3]{3}$.

9. [Putnam **2007 B4**] Let n be a positive integer. Find the number of pairs (P, Q) of polynomials with real coefficients such that
$$P(x)^2 + Q(x)^2 = x^{2n} + 1$$
and $\deg(P) > \deg(Q)$.