LSU Problem Solving Seminar - Fall 2016
Nov. 30: Putnam Review / Miscellaneous

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Website: `www.math.lsu.edu/~mahlburg/teaching/2016-Putnam.html`

Putnam Mathematical Competition, Sat., Dec. 3
Lockett Hall 244, 8:30 A.M. – 5:00 P.M.

Test-taking tips:

- **Format.** The Exam is given in two 3-hour sessions of 6 problems each, with a lunch break from 12:00 – 2:00 P.M. The morning session's problems are labeled **A1** − **A6**, and the afternoon's **B1** − **B6**.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of 120. Typically there is very little partial credit given, and a submitted problem will receive **0, 1, 2, 9,** or **10** points.
- **A1/A2/B1.** In recent years these three problems have been the "easiest" part of the exam. More generally, the problems in each session are roughly ordered by difficulty. This is not an absolute rule, but you should expect that **A1** will have a relatively short solution, whereas **A6** will not. You should plan on spending at least 15 minutes each trying to make any progress on **A1, A2, B1** before moving on to the rest of the Exam.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending approximately one hour writing your solution. Remember, it is better to solve one problem completely than several problems partially.

Main Problems

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1. The *floor function* (or "integer-part") is defined for real x by
$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}.$$
For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 12 \rfloor = 12$.
   (a) What is $\lfloor -1.8 \rfloor$?
   (b) Show that $2\lfloor x \rfloor \neq \lfloor 2x \rfloor$ in general.
   (c) Prove that $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$.
   (d) The *ceiling* function is defined similarly as the least integer larger than x, so
$$\lceil x \rceil := \min\{n \in \mathbb{Z} \mid x \leq n\}.$$
   Prove that $\lceil x \rceil = -\lfloor -x \rfloor$.

2. [Putnam **2005 B1**] Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.

3. Recall that the *Fibonacci numbers* are defined by $F_1 = 1, F_2 = 1$ and for $n \geq 3$,
$$F_n = F_{n-1} + F_{n-2}.$$

(a) Find all n such that F_n is a multiple of 5.

(b) Prove that there is some n such that F_n is a multiple of 2016.

4. Suppose that a, b, and c are the side lengths of a triangle, order so that $a \leq b \leq c$. Prove that the triangle is acute if and only if $a^2 + b^2 > c^2$.

5. [Putnam **2012 A1**] Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

6. (a) Let V be a collection of points on a sphere, that are *connected* by edges E. In particular, an edge joins two distinct points in V, and the property of connectedness means that there is a path along a sequence of edges joining **any** two points. The edges divide the sphere into distinct regions, or "faces" F. The *Euler characteristic* of such a configuration is defined to be

$$\chi := \#V - \#E + \#F,$$

where $\#V$ indicates the number of vertices, etc. It is a remarkable fact that this is a universal invariant: $\chi = 2$ for **all** configurations. For example, if V consists of three points that are joined as a triangle, then $\chi = 3 - 3 + 2 = 2$. Prove this fact.

(b) A *Platonic solid* is a polyhedron whose faces are all identical regular polygons (i.e. equilateral triangles, squares, pentagons with five equal sides, etc.). Use the Euler characteristic to identify and classify the **five** Platonic solids; in particular, how many edges and vertices does each have?

- A *tetrahedron* has 4 triangular faces;
- A *cube* has 6 square faces;
- An *octahedron* has 8 triangular faces;
- A *dodecahedron* has 12 pentagonal faces;
- An *icosahedron* has 20 triangular faces.

Remark: It is also possible to use the Euler characteristic to show that these are the only *Platonic solids.*

7. [Putnam **2013 A1**] Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

8. Determine all polynomials $f(x)$ that satisfy $f'(n) = n$ for all integers n.

Hint: Consider the function $g(x) := f(x+1) - f(x)$.

9. [Putnam **2010 A2**] Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.