- Virginia Tech Mathematics Contest. Sat., Oct. 22. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 3. **Sign-up deadline: Oct. 7**.

LSU Problem Solving Seminar - Fall 2016
Sep. 7: Invariants

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Website: `www.math.lsu.edu/~mahlburg/teaching/2016-Putnam.html`

Warm Up

1. A classroom begins the hour containing 6 people. Each minute either one person leaves, or three people enter; this continues for exactly one hour.
   (a) Find a sequence of moves that result in 70 people in the room at the end of the hour.
   (b) Is it possible to end with 100 people?
   Hint: Notice that after three minutes, there could be 3, 7, 11 *or* 15 *people...*

2. (a) Consider an 8×8 checkerboard with two corners removed. Your goal is to cover the remaining squares exactly with 1×2 dominoes (in any orientation). Prove that this is possible if the two corners are **not** diagonally opposite.
   (b) Now consider a 5×5 checkerboard with one square removed, which could be anywhere on the board. For which positions of the missing square is it possible to cover the remaining board with 1×3 tiles?
   Hint: Label each square in the first and fourth row by 1*, the second and fifth row by* 2*, and the third row by* 0*. Observe that any tile now covers three squares that sum to a multiple of* 3*. If you add up all of the tiles, what can you conclude about the missing square?*

Main Problems

3. You play a game in which 100 coins are initially placed on a table, and are randomly divided into groupings, with no restrictions on the size of each group or number of groups. You now make a series of moves, where each move consists of moving one coin from some group to another group that is **at least** as large. Thus it is legal to move a coin from a group of 9 coins to another group of 9 coins, but it is illegal to move a coin from a group of 4 coins to a group with 3 coins.

   What are the possible outcomes of this game? Can it go on forever?

   Hint: Let C_1 denote the number of coins in the largest group, let C_2 denote the total number of coins in the largest and second-largest groups, and for any i, let C_i denote the total number of coins in the i largest groups. What is the effect of a move on the values of the C_is?

4. [Gelca-Andreescu **71**] On an arbitrarily large chessboard, consider a generalized knight that can jump p squares in one direction and q in the other, $p, q \geq 0$. Show that such a knight can return to its initial position only after an even number of jumps.

   Hint: Call the starting position $(0, 0)$*, and let* (x_n, y_n) *be the position after* n *moves. If* p *is even and* q *is odd, what can you say about* $x_n + y_n$*?*

5. In a certain kingdom each soldier carries a colored shield, and wears his or her hair long and flowing so it is visible from a distance. Tradition requires that each soldier be uniquely identifiable, so that no two soldiers may have the same combination of shield color and hair color. For example, if one soldier has Brown hair and a Green shield, then any other soldier with Brown hair must have a non-Green shield. There are k shield colors and ℓ hair colors. Each year a series of competitions are held to determine an honorary Champion.

   First, the soldiers are grouped by shield color, and the strongest soldier is chosen from each grouping; let S_1 be the strongest soldier with shield color 1, and so on, so that S_k is the **strongest** soldier with shield color k. Then the Shield Champion S is chosen as the **weakest** soldier among $S_1, \ldots, S_k$. Second, the soldiers are grouped by hair color, and the weakest soldier is chosen from each grouping. If H_i denotes the **weakest** soldier with hair color i, then the Hair Champion H is the **strongest** soldier among $H_1, \ldots, H_\ell$.

   Finally, the overall Champion is the stronger of S and H; in the case of a tie, then S is declared the Champion.

   (a) Suppose that the kingdom has three soldiers, where Soldier 3 is stronger than Soldier 2, and Soldier 1 is the weakest. Soldier 3 has Black hair and a Red shield, Soldier 2 has Brown hair and a Red shield, and Soldier 1 has Brown hair and a Green shield. Determine S, H, and the overall Champion.

   (b) Find an example where S is the Champion.

   (c) Now suppose that the collection of soldiers is maximal. Because of the uniqueness condition, this means that there are $k\ell$ soldiers, with one soldier for each possible shield and hair color configuration. For example, in part (a) this would require adding a fourth soldier with Black hair and a Green shield. Prove that in this case the Champion is always S.

6. An apprentice clockmaker excitedly approaches his master early one afternoon, saying "Boss! I have a great idea to reduce our costs – instead of using two different sizes for minute and hour hands, we can just use the same pieces for each." But rather than being pleased by his assistant's cleverness, the clockmaker responds angrily "You fool! Look at your prototype; how am I supposed to know what time it is?"

   Prove that 1 o'clock and $10\frac{70}{143}$ minutes and 2 o'clock and $5\frac{125}{143}$ minutes are an **ambiguous** time pair. This means that if the hour and minute hands are the same length, then the two times are indistinguishable (assuming *continuously* moving hands). For an extra challenge, determine how many ambiguous time pairs occur during the course of a day!

7. [Putnam **1954 A1**] Let $n > 1$ be an odd integer. Let A be an $n \times n$ symmetric matrix each of whose rows and columns is a permutation of the numbers $1, 2, \ldots, n$. Show that the main diagonal of A is also a permutation of the numbers $1, 2, \ldots, n$.