- Virginia Tech Mathematics Contest. Sat., Oct. 22. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 3. **Sign-up deadline: Oct. 7**.

LSU Problem Solving Seminar - Fall 2016
Sep. 14: Calculus

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Useful facts:

- **Intermediate Value Theorem.** Suppose that $f(x)$ is a continuous function defined on the interval $[a, b]$, and r is a value in between $f(a)$ and $f(b)$, so that
$$f(a) < r < f(b) \quad \text{or} \quad f(a) > r > f(b).$$
Then there is some point c in the interior of the interval, $a < c < b$, such that $f(c) = r$.
In other words, a continuous function cannot "skip" any values.
- **Mean Value Theorem.** Suppose that $f(x)$ is differentiable on the interval $[a, b]$. Then there is a point $a < c < b$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
In other words, a differentiable function must achieve its **average** *slope at some point.*
- **Critical Points.** If $f(x)$ is a differentiable function on an interval $[a, b]$, then its maxima/minima must occur at the end points or the **critical points**, where are those x such that $f'(x) = 0$. The maxima/minima are classified by the negativity/positivity of $f''(x)$.
- **L'hospital's Rule.** Suppose $f(x)$ and $g(x)$ are differentiable. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an **indeterminate form** (i.e., $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
- **Ordinary Differential Equations.** A differential equation of the shape
$$f'(x) = g(x) \cdot f(x)$$
has a solution $f(x) = e^{\int g(x)dx}$, where $\int g(x)dx$ denotes an antiderivative of g.

Warm Up

1. You are given 100 feet of fencing with which to construct a rectangular pen for grazing along a river. In particular, the river will serve as one of the sides of the pen. What is the greatest possible area that can be enclosed?
2. Calculate the following limits.

   (a)
$$\lim_{x\to 1} \frac{x^3 + 2x - 3}{\sin(x - 1)},$$

(b)

$$\lim_{x \to 0} \frac{\left(e^{3x} - 1\right)\left(e^{2x} - 1\right)}{x^2}.$$

3. (a) Prove that $e^x \geq 1 + x$, with equality only when $x = 0$.
   Hint: Let $f(x) := e^x - 1 - x$, and determine the critical points.
   (b) Conclude that for any $a > 0$ and $x \neq 0$,
   $$a^{-x} > 1 - x \ln(a).$$

4. (a) Determine all differentiable functions $f(x)$ that satisfy $f'(x) = 3 \cdot f(x)$.
   (b) Find a function $f(x)$ such that $f'(x) = \frac{2}{x} \cdot f(x)$ for $x > 0$.

Main Problems

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5. Find the minimum value of $f(x) = x^2 + 2x - 3 + \dfrac{32}{x^3}$ over all positive x.

6. (a) Let $f(x)$ be a continuous function with domain $[0, 1]$ and range $[0, 1]$ (f is not necessarily onto its range). Prove that there is a **fixed point**, i.e., an $x \in [0, 1]$ such that $f(x) = x$.
   Hint: Let $g(x) := f(x) - x\ldots$
   (b) Find a fixed point if $f(x) = 1 - x^2$.

7. (a) You are given 100 feet of fencing to construct a garden in the shape of an isosceles triangle. What is the greatest possible area that can be enclosed?
   (b) You are again given 100 feet of fencing, this time to construct a garden in the shape of a right triangle. What is the greatest possible area that can be enclosed?

8. Find an integer n such that
   $$\lim_{x \to \frac{1}{2}} \frac{\big(\sin(\pi x)\big)^2 - 1}{(2x - 1)^n}$$
   exists and is non-zero. Determine the limiting value.

9. (a) Prove that there are no solutions to $xe^y = ye^x$ where x and y are distinct real values greater than 1.
   Hint: Let $f(x) := \dfrac{e^x}{x}$, and show that f is strictly increasing for $x > 1$.
   (b) [Gelca-Andreescu **421**] Prove that there are no positive numbers x and y such that
   $$x2^y + y2^{-x} = x + y.$$

10. [VTRMC **1993 # 3**] Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$, for $n = 1, 2, \ldots$. Prove that $f'(n) = 1$ and $f_n''(1) = 2$, for all $n \geq 2$.

11. [Putnam **1988 A2**] A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f' \cdot g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).