- Virginia Tech Mathematics Contest. Sat., Oct. 22. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 3. **Sign-up deadline: Oct. 3**.

LSU Problem Solving Seminar - Fall 2016
Sep. 21: Enumeration

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Useful facts: (n and k are non-negative integers)

- **Pigeonhole Principle.** If more than n objects are distributed among n sets, then some set contains multiple objects. **(Advanced Version.)** If more than nk objects are distributed among n sets, then one contains more than k objects.
- **Permutations.** The number of ordered lists of k distinct elements chosen from a set of n objects is $P(n,k) := \frac{n!}{(n-k)!}$.
- **Binomial Coefficients.** Given two non-negative integers n and k, the number of ways of choosing k (unordered) objects from a set of n is $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ (this is read as "n choose k"). They satisfy the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- **Binomial Theorem.** For an integer $n \geq 0$, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.
- **Number of subsets.** There are 2^n distinct subsets of a set with n elements.

Warm Up

1. (a) There are approximately 5000 first names in common usage in the U.S.*, and over $37,000$ students, staff, and faculty at LSU. Show that there is some group of least 8 students with the same first name.

   (b) Although there are many more last names used in the U.S. (possibly millions!), the 40,000 most popular last names account for over 80% of the population. The U.S. population is currently more than $320,000,000$. Show that there must be at least two people with the same name.

   ∗ Data from U.S. Census. In fact, the names James, John, Robert, Michael, William, David, and Mary each account for more than 1% of U.S. citizens!

2. A pizza parlor offers 7 toppings: Anchovies, Bacon, Chicken, Dried Tomatoes, Eggplant, Feta Cheese, and Garlic. How many distinct choices are there for each of the following menu options?

   (a) The *Garlic Bread Sampler* consists of 3 small loaves of garlic bread, with a choice of 1 topping for each. Each loaf is prepared differently: the first has cheese, the second has tomato sauce, and the third has oil.

(b) An *Italian Sub* consists of 4 distinct toppings layered on a sandwich. Since the upper ingredients will drip onto the lower, the order matters!

(c) A *Standard Pizza* comes with your choice of $1--3$ distinct toppings; note that they are spread out on the pizza, so the order does not matter.

(d) A *Loaded Pizza* consists of any 4 toppings (with repeats allowed), plus an additional choice of Butter, Garlic, Oil, or Parmesan crust.

3. If $n \geq 0$ is an integer, evaluate the sum

$$2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + 2\binom{n}{n-1} + 1.$$

Main Problems

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4. A committee is to be chosen from a group of 11 students. How many possible committees are there under the following restrictions:

   (a) The committee consists of 3 students;

   (b) The committee contains an odd number of members;

   (c) The committee contains at most 5 members.

5. (a) Show that if 5 points are placed in a square of side length 2, then there must be two points within a distance of $\sqrt{2}$ of each other. Is this the best possible bound?

   (b) [Gelca-Andreescu **44**] Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

   The bound suggests a decomposition into unit squares.

6. [Gelca-Andreescu **871**] Prove that

$$\binom{n}{k} + \binom{n+1}{k} + \binom{n+2}{k} + \cdots + \binom{n+m}{k} = \binom{n+m+1}{k+1} - \binom{n}{k+1}.$$

7. [VTRMC **2008 # 2**] How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $(1,3,3,3,3,3)$ and $(1,3,1,3,1,3,1,3)$.)

8. [Putnam **1978 A1**] Let A be any set of at least 20 distinct integers chosen from the arithmetic progression $1, 4, 7, 10, \ldots, 100$. Prove that there must be two distinct integers in A that sum to 104.