- Virginia Tech Mathematics Contest. Sat., Oct. 22. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 3. **Sign-up deadline: Oct. 3**.

LSU Problem Solving Seminar - Fall 2016
Sep. 28: Geometry

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- **Law of Cosines.** If a triangle has sides of lengths a, b, and c, and α is the angle opposite the side of length a, then
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
- **Law of Sines.** If β is the angle opposite b, and γ is the angle opposite c, then
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R},$$
where R is the radius of the circumscribed circle (which contains the vertices of the triangle).
- For all x,
$$\sin^2(x) + \cos^2(x) = 1,$$
- **Addition Formulas.** For all x and y,
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
- **Heron's Formula.** If a triangle has side lengths a, b, and c, then its area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s := \dfrac{a+b+c}{2}$ is the *semiperimeter.*

Warm Up

1. Let O be the center of two concentric circles, of radii 3 and 6. A line from O hits the outer circle at the point B, and the inner circle at the point A. There is another point C on the outer circle such that $|AC| = 5$. Find $|BC|$.
2. It is a general fact that the sum of trigonometric functions can be expressed as a **single** trigonometric function (possibly with an angular *phase* shift and *amplitude* shift). Find constants α, β such that
$$\sin(x) + \cos(x) = \alpha\sin(x+\beta).$$

Main Problems

3. Two rectangles are glued together to form an "L" shape, without overlap. In other words, if one rectangle has vertices at $(0,0),(a,0),(0,b)$, and (a,b), then the second rectangle has vertices $(a,0),(c,0),(a,d),(c,d)$ with $d \leq b$. Describe how to make **one** straight cut that gives two pieces of equal area.

   Hint: An incorrect answer is to find the center of each rectangle and cut along the line joining these two points. This results in **three** *pieces, although it is true that the corner piece has the same area as the other two combined.*

4. Suppose that ABC is a triangle. A *median* is a line segment from a vertex to the midpoint of the opposite side; for example, one median joins A to the midpoint of $\overline{BC}$.

   (a) Prove that the three medians intersect in a single point.

   Remark: This point is known as the **centroid** *of the triangle – which is also the center of mass!*

   (b) If you draw the three medians, ABC is divided into 6 smaller triangles. Prove that they all have the same area.

5. (a) A cube has vertices at $(0,0,0)$ and $(1,7,5)$, and all of its vertices have integer coordinates. Determine the coordinates of **all** of its vertices.

   (b) (From [Gelca-Andreescu **632**]) Prove that if the vertices of a cube have integer coordinates, then the length of the edge of the cube is an integer.

6. If ABC is a triangle, the *circumscribed circle* is the (unique) circle that passes through A, B and C.

   (a) Suppose that a triangle has side lengths $(3,4,5)$. Find the diameter of the circumscribed circle.

   (b) Let R denote the area of ABC. Prove that in general, the diameter of the circumscribed circle is given by the formula

$$d = \frac{|\overline{AB}| \cdot |\overline{AC}| \cdot |\overline{BC}|}{2R}.$$

7. [VTRMC **1987 # 2**] A triangle with sides of lengths a, b, and c is partitioned into two smaller triangles by the line which is perpendicular to the side of length c and passes through the vertex opposite that side. Find *integers* $a < b < c$ such that each of the smaller two triangles is similar to the original triangles and has sides of integer lengths.

8. Suppose that ABC is a triangle. An *inscribed* rectangle is contained in ABC and intersects each of the three edges. In other words, one side of the rectangle is part of an edge of ABC, and the other two vertices of the rectangle are on the other two edges. Prove that the area of an inscribed rectangle is at most **half** the area of ABC.

9. [Putnam **1985 A2**] Let ABC be an acute triangle. Inscribe a rectangle R with one side along $\overline{BC}$. Then inscribe a rectangle S in the triangle formed by R, $\overline{AB}$, and $\overline{AC}$. Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\dfrac{A(R)+A(S)}{A(T)}$, where ABC ranges over all triangles and R, S over all rectangles as above.