- Virginia Tech Mathematics Contest. Sat., Oct. 22.

LSU Problem Solving Seminar - Fall 2016
Oct. 5: Integration

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- **Partial Fractions.** If $f(x)$ is a polynomial whose degree is less than n, then there are constants $a_1, \ldots, a_n$ such that
$$\frac{f(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{x-r_1} + \cdots + \frac{a_n}{x-r_n}.$$
(Here the roots r_i must be distinct – there is a more complicated version for repeated roots.)
- **Fundamental Theorem(s) of Calculus.** Suppose that $f(x)$ is a continuous function.
  - If $F(x)$ is an antiderivative of f, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.
  - Define $g(x) := \displaystyle\int_a^x f(t)dt$. Then $g'(x) = f(x)$.
- **Integration By Parts.** Suppose that f and g are differentiable. Then
$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$
- **Substitution.**
$$\int_{x=a}^{b} f(u(x))u'(x)dx = \int_{u=u(a)}^{u(b)} f(u)du.$$
- **Symmetries and Substitution.** Remember, integration problems on Mathematics Contests are meant to have solutions! A complicated integral often has a hidden symmetry or substitution that makes it much easier to evaluate. For example, if $f(x)$ is an *odd* function, then $\int_{-a}^{a} f(x)\, dx = 0$.

Warm Up

1. Calculate the following antiderivatives (indefinite integrals):
   (a) $\displaystyle\int \cos(3x)\, dx$,
   (b) $\displaystyle\int \frac{x}{x^2+1}\, dx$
   (c) $\displaystyle\int \frac{x^2}{x^2-3x+2}\, dx$,
   Hint: Try partial fractions.
2. Evaluate the following integrals without doing any serious computation:

(a) $\int_0^3 \sqrt{9-x^2}\, dx.$

(b) $\int_{-1}^1 \frac{\sin(x)}{\cos(2+x)+\cos(2-x)}\, dx.$

3. Find a linear function $f(x) = ax + b$ such that

$$\int_0^1 f(x)dx = 1 \qquad \text{and} \qquad \int_0^1 xf(x)dx = 1.$$

Main Problems

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4. Find the antiderivative

$$\int \left(1+3x^3\right) e^{x^3} dx.$$

Hint: What is the derivative of e^{x^3}*? Regroup terms around that....*

5. (a) What is the area of a triangle with vertices $(0,0), (1,0)$, and $(1,1)$?
   (b) What is the volume of a tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$?
   (c) What is the volume of a tetrahedron with vertices $(0,0,0), (1,0,0), (1,1,0)$ and $(1,1,1)$?

6. (a) Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{e^{2x}-e^{\frac{2}{x}}}{x} dx.$$

Hint: Let I *denote the integral. Make the substitution* $x = \frac{1}{t}$*, and show that* $I = -I$*.*

   (b) (From [Gelca-Andreescu **460**]) Evaluate

$$\int_0^\infty \frac{\ln(x)}{x^2+1} dx.$$

Hint: Try the substitution $x = \frac{1}{t}$*.*

7. (a) Evaluate the integral

$$\int_0^1 \left(x^2+\sqrt{x}\right)\, dx.$$

   (b) Sketch the graphs of the functions x^2 and $\sqrt{x}$, and reprove your answer from part (a) without doing any computation.
   (c) Suppose that $f(x) : [0,a] \to [0,a]$ is a continuous, increasing function that satisfies $f(0) = 0$ and $f(a) = a$. Show that

$$\int_0^a \left(f(x)+f^{-1}(x)\right)\, dx = a^2.$$

Hint: Draw the right picture, and the problem becomes geometric ...

8. [VTRMC **2005 # 6**] Compute $\int_0^1 \left((e-1)\sqrt{\ln(1+ex-x)}+e^{x^2}\right)dx$.

9. [Putnam **1991 A4**] Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

   - The c_i have no limit point in the finite plane,
   - The sum of the areas of the D_i is finite, and
   - Every line in the plane intersects at least one of the D_i?