- Virginia Tech Mathematics Contest. Sat., Oct. 22, 8:30 – 11:30, Coates 225.

LSU Problem Solving Seminar - Fall 2016
Oct. 12: Number Theory

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Useful facts:

- **Divisibility Tests.** A positive integer n is divisible by:
  - 2 if its last digit is a multiple of 2;
  - 3 if the sum of its digits is a multiple of 3;
  - 4 if its last two digits are a multiple of 4;
  - 5 if its last digit is 0 or 5;
  - 7 if ______________________________
  - 9 if the sum of its digits is a multiple of 9;
  - 11 if the alternating (with plus and minus signs) sum of its digits is a multiple of 11.
- The prime factorization of the current year is $2016 = 2^5 \cdot 3^2 \cdot 7$, and the previous year was $2015 = 5 \cdot 13 \cdot 31$.
- **Fermat's Little Theorem.** If p is prime and a is any integer, then $a^p - a$ is a multiple of p.

  Advanced version (**Lagrange's Theorem.**) If n is an integer and a has no common prime factors with n, then $a^{\phi(n)} - 1$ is a multiple of n (see below for Euler's ϕ-function).
- **Remainders of Squares.** For any integer n, n^2 can only have the following remainders:

  0 or 1 when divided by 3; 0 or 1 when divided by 4; 0, 1, 4, 5, 6, or 9 when divided by 10.
- **Modular/Remainder Arithmetic.** Write $a \bmod m$ to represent the *remainder* when a is divided by m. Then

$$(a+b) \bmod m = \big(a \bmod m + b \bmod m\big) \bmod m,$$
$$(ab) \bmod m = \big(a \bmod m \cdot b \bmod m\big) \bmod m.$$

  For example, to determine the remainder when $22 \cdot 31 + 50$ is divided by 7, calculate

$$22 \cdot 31 + 50 \equiv 1 \cdot 3 + 1 \equiv 3 + 1 \equiv 4 \pmod 7.$$

  Compare this to the "direct" answer: $22 \cdot 31 + 50 = 682 + 50 = 732 = 104 \cdot 7 + \boxed{\mathbf{4}}$.
- **Greatest Common Divisors.** If a and b are integers with prime factorizations $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, b = p_1^{\beta_1} \cdots p_r^{\beta_r}$, their greatest common divisor is

$$\gcd(a,b) = p_1^{\min\{\alpha_1,\beta_1\}} \cdots p_r^{\min\{\alpha_r,\beta_r\}}.$$

  The equation $ax + by = N$ has integer solutions if and only if $\gcd(a,b)$ divides N.
- **Euler's Totient Function.** If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then define

$$\phi(n) := n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

  Among the integers $1, 2, \ldots, n$, exactly $\phi(n)$ of them satisfy $\gcd(a,n) = 1$. Furthermore, if $\gcd(a,n) = 1$, then $a^{\phi(n)}$ has remainder 1 when divided by n.

- **Casting Out Nines.** If n is an integer and s is the sum of its decimal digits, then $n - s$ is a multiple of 9.

Warm Up

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1. (a) Is 4365 a multiple of 9? Of 11?
   (b) Is 26061 a multiple of 3? Of 7?

2. Find the prime factorization of 1001.

3. (a) Determine the remainder when $11 \cdot 22 \cdot 33$ is divided by 13.
   (b) Determine the last two digits of 2016^{2016}.
   (c) Determine the remainder when 2014^{2016} is divided by 2015.

Main Problems

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4. A *Pythagorean triple* consists of positive integers (a, b, c) such that $a^2 + b^2 = c^2$ (geometrically, such a triple describes a right triangle with integer side lengths). For example, $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples. Find **all** Pythagorean triples of the form $(a, n, n + 1)$.

   Remark. There are certainly other Pythagorean triples, such as $(8, 15, 17)$ or $(20, 21, 29)$.

5. Find all integers x and y such that $x + y \geq \dfrac{xy}{2}$.

6. Suppose that $n, k \geq 1$ are integers. Prove that
$$\frac{n \cdot (n+1) \cdot (n+2) \cdots (n+k-1)}{1 \cdot 2 \cdot 3 \cdots k}$$
   is an integer.

7. (a) What is the least common multiple of $1, 2, 3, 4, 5, 6, 7, 8, 9$, and 10?
   (b) Find the smallest positive integer n such that n has remainder 1 when divided by 2, remainder 2 when divided by 3, continuing to remainder 9 when divided by 10.

   Hint: There is such an integer: for example, $10! - 1 = 3628799$ *has the required remainders, but this is not the smallest!*

8. (a) Prove that $7^7 + 8^8$ is a multiple of 11.
   (b) One of the following sums is a multiple of 2016 – determine which one!
$$87^{87} + 63^{63} \qquad \text{or} \qquad 111^{111} + 81^{81}?$$

9. (a) Prove that $x^2 + 5y^2 = z^2$ has infinitely many nontrivial solutions (a solution is *trivial* if at least one of x, y and z is zero).

(b) ([Gelca-Andreescu **707**]) Prove that the system of equations

$$\begin{aligned} x^2 + 5y^2 &= z^2 \\ 5x^2 + y^2 &= t^2 \end{aligned}$$

does not have nontrivial integer solutions.

Hint: Consider the remainders when divided by 3.

10. [VTRMC **1999 # 6**] A set S of distinct positive integers has property ND if no element x of S divides the sum of the integers in any subset of $S \setminus \{x\}$. Here $S \setminus \{x\}$ means the set that remains after x is removed from S.

   (i) Find the smallest positive integer n such that $\{3, 4, n\}$ has property ND.

   (ii) If n is the number found in (i), prove that no set S with property ND has $\{3, 4, n\}$ as a proper subset.

11. Consider the sequence of powers of 3 :,

$$b_0 = 1,\ b_1 = 3,\ b_2 = 3^2,\ \ldots, b_n = 3^n,\ \ldots.$$

Determine all n such that the last two digits of b_n are 43.

12. [Putnam **1985 A4**] Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?