- Virginia Tech Mathematics Contest: Sat., Oct. 22, 8:30 – 11:30, Coates 225.
- **Format.** This written exam consists of 7 problems to be solved in 2 hours and 30 minutes.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of **70**. The grading is strict, with very little partial credit given for guesses or incomplete proofs.
- **Unordered.** The problems are **not** ordered by difficulty, so you should plan on spending the first 15–20 minutes reading all of the problems and then deciding which ones you are best able to answer.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending up to one hour writing your solution. Remember, it is better to solve one problem completely than several problems partially.

LSU Problem Solving Seminar - Fall 2016
Oct. 19: Probability

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Website: `www.math.lsu.edu/~mahlburg/teaching/2016-Putnam.html`

Useful facts:

- **Probability Spaces.** A (countable) probability space consists of a set of distinct events $A_1, A_2, \ldots$ and a probability function $0 \leq p \leq 1$ such that $p(A_1) + p(A_2) + \cdots = 1$.

  In a finite probability space, typically $p(A) = \dfrac{\#\text{ outcomes in } A}{\#\text{ total outcomes}}$. For example, if two dice are rolled, there are 5 ways to obtain a sum of 6 (namely, $5+1, 4+2, 3+3, 2+4, 1+5$), and $6^2 = 36$ total combinations, so $p(6) = \frac{5}{36}$.

- **Random Variables and Expectation.** A random variable X assigns a real value x to each event A. The expected value, or *average* of X is

$$E[X] := \sum_x x \cdot P(X = x).$$

  For example, the expected number of Tails when two coins are flipped is $0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} = 1$.

- **Additivity of Expectation.** If X and Y are random variables, $E[X+Y] = E[X] + E[Y]$.

- **Exponential and Stirling approximation.** Use the following formulas to approximate discrete probabilities for large n (and small k):

$$\left(1 - \tfrac{1}{n}\right)^n \sim e^{-1}, \qquad n! \sim \left(\tfrac{n}{e}\right)^n, \qquad \text{and} \qquad \tbinom{n}{k} \sim \tfrac{n^k}{k!}.$$

Warm Up

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1. At LSU, 88% of students like Gumbo, 72% Jambalaya, and 51% like Boiled Crawfish*. Prove that there is at least one student who enjoys all three regional dishes.

* Statistics may not be accurate.

2. (a) What is the probability that among n people no one shares a birthday?
   (b) What is the probability that if you meet 730 people, none of them will share **your** birthday? Find a simple approximation of your answer in terms of e.

3. (a) What is the expected total when two dice are rolled?
   (b) What is the expected total when three dice are rolled?

Main Problems

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4. A standard deck of 52 cards is thoroughly shuffled.
   (a) What is the probability that the first card is an Ace?
   (b) What is the probability the last card is a Spade?
   (c) What is the probability that the first card is an Ace and the last card is a Spade?
   (d) What is the probability that all four Aces are adjacent in the deck?

5. A point P is picked randomly in the Cartesian plane, and a circle of radius 1 is drawn around P. We say that (x, y) is a *lattice point* if x and y are integers.
   (a) What is the probability that the circle contains **exactly** 2 lattice points?
   (b) What is the probability that the circle contains at least 3 lattice points?

6. A fair coin is flipped repeatedly. What is the probability that you see:
   (a) A Head before a total of two or more Tails?
   (b) At least two Heads before the first Tail?
   Hint: The first two parts can be solved by considering the first several flips.
   (c) Two consecutive Heads before two consecutive Tails?
   Hint: This is also easy!

7. A fair coin is flipped repeatedly. What is the probability that you see:
   (a) [From Gelca-Andreescu **917**] Three consecutive Heads before two consecutive Tails?
   Hint: Focus on the Heads event; let P_H be the probability that three Heads occur first if the first flip is H, let P_T be the probability that three Heads occur first if the first flip is T, and similarly for $P_{HT}, P_{TH}, \ldots$. Now find relations between these probabilities, using the fact that $P_{HHH} = 1$, and $P_{TT} = 0$.
   (b) The (consecutive) sequence HTH before HHT?

8. A fair coin is flipped repeatedly.
   (a) What is the expected number of flips until you see the sequence HTH for the first time?
   (b) What is the expected number of flips until you see the sequence HHT for the first time?

9. The first two parts of this problem illustrate a commonly known *Investor's Paradox.* Each year an investor will gain 30% or lose 25% with equal probability.

(a) Show that the investor's expected return is positive.

(b) Show that if $n \geq 2$, then the majority of the time the investor will have lost money after n years.

(c) [VTRMC **2003 # 1**] An investor buys stock worth \$10,000 and holds it for n business days. Each day he has an equal chance of either gaining 20% or losing 10%. However, in the case he gains every day (i.e. n gains of 20%), he is deemed to have lost all his money, because he must have been involved with insider trading. Find a simple formula for the amount of money he will have on average at the end of the n days.

10. [Putnam **2002 B1**] Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. what is the probability she hits exactly 50 of her first 100 shots?