- Virginia Tech Mathematics Contest. Sat., Oct. 21. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 2. **Sign-up deadline: Oct. 6**.

LSU Problem Solving Seminar - Fall 2017
Aug. 23

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Warm Up

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1. (a) Split the integers $1, 2, 3, \ldots, 20$ into two groups so that the **sum** of all numbers in each group is the same.

   (b) Can you do the same for the integers $1, 2, \ldots, 10$?

   (c) Can you split the integers $1, 2, \ldots, 20$ into two groups with the same **product**?

2. Suppose that in car factory A 10 workers complete 120 cars during a 4-hour shift, and in Factory B 5 workers complete 100 cars in an 8-hour shift. If the following day both factories work on the same model of car for 6 hours, with 6 workers in Factory A and 10 in Factory B, how many cars will they paint in total?

Main Problems

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3. You are given three piles of coins, and are allowed to perform "doubling" moves from one pile to another. This means that you may move coins from a larger pile to a smaller one so that the size of the smaller pile is **doubled**. For example, if you had piles of size $(12, 7, 8)$, you could move 7 coins from the first to second pile, resulting in piles of size $(5, 14, 8)$; you could also move 8 coins from the first to the third, giving $(4, 7, 16)$. The goal of the game is to obtain three piles with an **equal** number of coins.

   (a) If you start with piles of $3, 4$, and 5 coins, find a sequence of moves that results in three piles of equal size.

   (b) Now suppose that the piles have $14, 7$, and 3 coins. Find a winning sequence of moves.

   (c) Show that if the piles have size $12, 7$, and 8, there is **no** sequence of moves that results in three piles of the same size.

4. The *outer edge* of a cone is the line from its vertex to its base that travels along the outer surface. What is the maximum volume among all cones with outer edge length 3?

5. A positive integer n is split into *near-halves* when it is written as $n = m_1 + m_2$ such that m_1 and m_2 differ by at most 1. For example, $20 + 20$ are near-halves of 40, and $20 + 19$ are near-halves of 39.

   (a) Show that there is exactly **one** way to split n into near-halves.

   (b) Suppose that n is split into near-halves $m_1 + m_2$, which are then themselves split into near-halves $m_1 = k_1 + k_2, m_2 = \ell_1 + \ell_2$. Show that this gives a splitting of n into near-quarters $n = k_1 + k_2 + \ell_1 + \ell_2$, where each of k_1, k_2, ℓ_1, ℓ_2 differ by at most 1. For example, $20 = 10 + 10$ and $19 = 10 + 9$, so overall we have the near-quarters $39 = 10 + 10 + 10 + 9$.

   (c) What happens if you continue to split the near-quarters into near-halves? How far can this procedure continue?

6. [Putnam **2003 A1**] Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$, there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$.

7. [Putnam **1988 A1**] Let R be the region consisting of the points (x, y) in the Cartesian plane that satisfy both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch R and find its area.