LSU Problem Solving Seminar - Fall 2017
Nov. 1: Number Theory

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Useful facts:

- **Divisibility Tests.** A positive integer n is divisible by:
  - 2 if its last digit is a multiple of 2;
  - 3 if the sum of its digits is a multiple of 3;
  - 4 if its last two digits are a multiple of 4;
  - 5 if its last digit is 0 or 5;
  - 7 if ____
  - 9 if the sum of its digits is a multiple of 9;
  - 11 if the alternating (with plus and minus signs) sum of its digits is a multiple of 11.
- The current year, 2017, is prime, and the previous year has prime factorization $2016 = 2^5 \cdot 3^2 \cdot 7$.
- **Fermat's Little Theorem.** If p is prime and a is any integer, then $a^p - a$ is a multiple of p.
- **Remainders of Squares.** For any integer n, n^2 can only have the following remainders:

  0 or 1 when divided by 3; 0 or 1 when divided by 4; 0, 1, 4, 5, 6, or 9 when divided by 10.
- **Greatest Common Divisors / Least Common Multiples.** If a and b are integers with prime factorizations $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, b = p_1^{\beta_1} \cdots p_r^{\beta_r}$, their greatest common divisor is

  $$\gcd(a,b) = p_1^{\min\{\alpha_1,\beta_1\}} \cdots p_r^{\min\{\alpha_r,\beta_r\}}.$$

  The least common multiple is found by replacing min by max.

  The equation $ax + by = N$ has integer solutions if and only if $\gcd(a,b)$ divides N.
- **Euler's Totient Function.** If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then define

  $$\phi(n) := n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

  Among the integers $1, 2, \ldots, n$, exactly $\phi(n)$ of them satisfy $\gcd(a,n) = 1$. Furthermore, if $\gcd(a,n) = 1$, then $a^{\phi(n)}$ has remainder 1 when divided by n.
- **Casting Out Nines.** If n is an integer and s is the sum of its decimal digits, then $n - s$ is a multiple of 9.

Warm Up

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1. Answer these questions without using a calculator!
   (a) Is 8294 a multiple of 9? Of 11?
   (b) Is 12873 a multiple of 3? Of 7?
2. Find the prime factorization of the following integers:

   (a) 101;
   (b) 399;
   (c) 224.

3. For each of the following sets of conditions, find a suitable n, or explain why none exist.
   (a) n has remainder 6 when divided by 7, and remainder 7 when divided by 8;
   (b) n has remainder 2 when divided by 8, and remainder 4 when divided by 12.

4. Find an $n \geq 1$ such that 2^n and 5^n begin with the same decimal digit.

Main Problems

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5. Prove that for all $n \geq 1$, $\dfrac{15n+2}{10n+1}$ is an irreducible fraction (i.e. numerator and denominator are already in lowest terms).

6. Determine the last two (decimal) digits of 2019^{2018}.

7. (a) What is the least common multiple of $4, 5, 6, 7,$ and 8?
   (b) Find the smallest positive integer n such that n has remainder 0 when divided by 4, remainder 1 when divided by 5, remainder 2 when divided by 6, remainder 3 when divided by 7, and remainder 4 when divided by 8.
   Hint: There is such an integer: for example, $8! - 4 = 40316$ *has the required remainders, but this is not the smallest!*

8. Find positive integers $n_1, n_2, \ldots, n_9$ that are **not** $1, 2, \ldots, 9$ such that their product is $n_1 n_2 \cdots n_9 = 362880 = 9!$, and their sum is $n_1 + \cdots + n_9 = 45$.
   Some of the n_i *may be in* $1, 2, \ldots, 9$*; the requirement is simply that the set of* n_i *must not be exactly* $\{1, 2, \ldots, 9\}$*. Repeated values among the* n_i *are allowed – in fact, they are required! (Why?)*

9. [Gelca-Andreescu **17**] Prove that for any positive integer n there is an n-digit number that is divisible by 2^n and contains only the digits 2 and 3. For example, 232 is a multiple of $2^3 = 8$.

10. (a) Suppose that an integer has the prime factorization $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where the p_i are distinct primes. Prove that the number of distinct positive divisors of n is
$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1).$$
    (b) Find the three-digit integer with the **greatest** number of divisors.
    It is easier to find a three-digit integer (not unique) with the least number of divisors – why?

11. [Putnam **1983 A1**] How many positive integers n are there such that n is an exact divisor of at least one of the numbers
$$10^{40}, 20^{30}?$$

12. [VTRMC **1995 # 7**] If n is a positive integer larger than 1, let $n = \prod p_i^{k_i}$ be the unique prime factorization of n, where the p_i's are distinct primes $2, 3, 5, 7, 11, \ldots$, and define $f(n)$ by $f(n) := \sum k_i p_i$, and $g(n)$ by $g(n) := \lim_{m\to\infty} f^m(n)$, where f^m means the m-fold application of f. Then n is said to have *property* H if $n/2 < g(n) < n$.

    (a) Evaluate $g(100)$ and $g(10^{10})$.

    (b) Find all positive odd integers larger than 1 that have property H.

13. [Putnam **2013 A2**] Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r overall such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

14. (a) Prove that if n is a positive integer greater than 1, then $2^n - 1$ is not a multiple of n.

    Hint: Suppose that p is a prime divisor of n. What does Fermat's Little Theorem imply about 2^n?

    (b) Find all positive integers n such that $2^n + 1$ a multiple of n.