LSU Problem Solving Seminar - Fall 2017
Nov. 8: Inequalities

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Useful facts:

- **Arithmetic-Geometric Mean Inequality.** If $a_1, \dots, a_n$ are non-negative real numbers, then
$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$
Furthermore, the right side is strictly larger than the left unless all of the a_i are equal.

- **Hölder's p-norm Inequality.** If $0 < p < q$ and $a_1, \dots, a_n$ are non-negative real numbers, then
$$\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}},$$
with strict inequality unless all of the a_i are equal.

- **Cauchy-Schwarz Inequality.** If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then
$$\left(a_1 b_1 + \cdots + a_n b_n\right)^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$
Furthermore, the right side is strictly larger unless $(b_1, \dots, b_n) = (\lambda a_1, \dots, \lambda a_n)$ for some real λ. Written in vector notation and Euclidean distance, $|\overrightarrow{a} \cdot \overrightarrow{b}|^2 \leq |\overrightarrow{a}|^2 \cdot |\overrightarrow{b}|^2$.

- **Triangle Inequality.** If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then
$$\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$$
Written in vector notation, $|\overrightarrow{a} + \overrightarrow{b}| \leq |\overrightarrow{a}| + |\overrightarrow{b}|$.

- **Rearrangement Inequality.** Suppose that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. If $a_1', a_2', \dots, a_n'$ is any reordering of $a_1, \dots, a_n$, then
$$a_1' b_1 + a_2' b_2 + \cdots + a_n' b_n < a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Warm Up

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1. For each of the following pairs, determine which expression is larger (without using a calculator!):

   (a) 6^{5^4} or 4^{5^6}? What about: 3^{3^3} or $2^{2^{2^2}}$?

   (b) $\sqrt{2000} - \sqrt{17}$ or $\sqrt{2001} - \sqrt{18}$?

   (c) $\left(1 + \frac{1}{2017}\right)^{2017}$ or $\frac{1}{\left(1 - \frac{1}{2017}\right)^{2017}}$?

2. (a) Suppose that a and b are positive real numbers. Show that
$$a^2 + b^2 \geq 2ab.$$

(b) If a and b are positive real numbers, what is the minimum value of

$$\frac{a}{b} + \frac{b}{a}?$$

3. (a) Suppose that a, b, and c are positive real numbers. Show that

$$ab^2 + bc^2 + a^2c \geq 3abc.$$

(b) If a, b, c are positive real numbers, what is the minimum value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a}?$$

Main Problems

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4. The following questions appeared on the Calculus problem sheet (Sep. 20). Use the Arithmetic-Geometric Mean Inequality to answer them **without** taking any derivatives!

   (a) Find the minimum value of $f(x) = x^2 + -2 + \frac{16}{x^2}$ over all positive x.

   (b) A rectangular area 400 square feet is to be fenced off. Find the dimensions that minimize the length of fencing used. In other words, find the minimum value of $2x+2y$ if $xy = 400$.

   (c) A reinforced shipping crate with volume 1500 cubic feet is to be built, where the bottom is doubly reinforced (this means that two layers of wood are used). What is the minimum amount of wood required? In other words, find the minimum value of $2h\ell + 2hw + 3\ell w$ if $h\ell w = 1500$.

5. Suppose that $x_1, \ldots, x_n$ are positive real numbers. Prove that either

$$x_1 + x_2 + \cdots + x_n \geq n \qquad \text{or} \qquad \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq n.$$

6. A box may be shipped at a standard flat rate so long as its dimensions are not too large: the *linear length*, which is defined to be the sum of the height, length, and width, cannot exceed 80 inches. Show that this system cannot be "cheated", as it is not possible to pack a box with larger volume inside one with larger linear length.

7. Suppose that $k \geq 0$. What is the maximum value of

$$f(x) = \frac{(x+k)^2}{x^2+1}?$$

Hint: Write the numerator term as $x + k = x \cdot 1 + 1 \cdot k$ and use the Cauchy-Schwarz inequality.

8. The *Harmonic mean* of positive real numbers $x_1, x_2, \ldots, x_n$ is

$$H(x_1, \ldots, x_n) := \left(\frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \right)^{-1}.$$

(a) A painter can finish 1000 square feet of wall in 2 hours, and her junior colleague can finish it in 3 hours. One week they are both assigned to a large project with a total of 20000 square feet to be painted. The first painter says "Since I can paint 1000 square feet in 2 hours, and you can do it in 3 hours, on average it takes one of us 2.5 hours. So if we split the project into two 10000 square foot parts (since there are two of us), the total time should be $10 \cdot 2.5 = 25$ hours."

The client was furious when he realized that he had been billed for 25 hours, as the job actually required less time – how much time was really needed?

(b) What does part (a) have to do with harmonic means?

(c) Prove the *Harmonic-Geometric Mean Inequality:*

$$H(x_1, \ldots, x_n) \leq \sqrt[n]{x_1 \cdots x_n},$$

with equality only when all x_i are equal.

Hint: Use the Arithmetic-Geometric Mean Inequality.

9. In this problem you will explore some of the surprising properties of exponential towers. Define $a_0 = b_0 := 1$, and for $n \geq 1$, $a_n := 2^{a_{n-1}}, b_n := 3^{b_{n-1}}$. Thus a_n is an exponential tower of n 2s: $2^{2^{\cdots^2}}$, and b_n is a similar tower of n 3s.

   (a) Show that $a_n < b_n$ for all $n \geq 1$ (this is easy!).

   (b) Show that $a_3 > 2b_1 + 2$.

   (c) If $a_n > 2b_{n-2} + 2$, show that $a_{n+1} > 2b_{n-1} + 2$.

   Remark: Thus $b_{n-2} < a_n < b_n$ for all n; in other words, a_n grows "nearly" as quickly as b_n, lagging by at most two terms. The shocking conclusion is that the growth of an exponential tower $x^{x^{\cdots^x}}$ depends much more on the **length** *than the value of the base x.*

10. [**Gelca-Andreescu 303**] Consider the sequences $(a_n)_n$ and $(b_n)_n$ defined by $a_1 := 3, b_1 := 100, a_{n+1} := 3^{a_n}, b_{n+1} := 100^{b_n}$. Find the smallest number m for which $b_m > a_{100}$.

11. (a) Prove that if $a_1 < a_2$ and $b_1 < b_2$, then

$$a_1 b_2 + a_2 b_1 < a_1 b_1 + a_2 b_2.$$

   Remark: This can be used to give an inductive proof of the Rearrangement Inequality.

   (b) A store is running a "Cash Grab" promotion in which a random winner gets to open a cash register and take out some bills. The registers hold bills in the denominations \$1, \$5, \$10, \$20, \$50, and \$100, and the lucky winner gets to take 6 bills from one denomination, 5 from another, and so on, with 4, 3, 2, and 1 bill(s), all from different denominations. How should the winner choose the bills?

12. [Putnam **1996 B3**] Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1.$$