Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 22 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Nov. 29.
- Putnam Mathematical Competition, **Sat., Dec. 2.** The Exam will take place in Lockett 244 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2017
Nov. 15: Polynomials

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be a polynomial with real coefficients. It is *monic* if the leading coefficient $a_n = 1$. The *degree* of a polynomial is the exponent of the leading term, in this case n. A *root* of f is a value r such that $f(r) = 0$.

- **Rational Roots Test.** If all of the a_i are integers and $r = \frac{p}{q}$ is a root, then p is a divisor of a_0 and q is a divisor of a_n.
- **Descartes' Rule of Signs.** If the non-zero coefficients of $f(x)$ change sign s times, then f has at most s positive roots (with multiplicity). The actual number of positive roots is less than s by some multiple of 2. Replacing x by $-x$ gives a similar test for negative roots.
- **Polynomial Division Algorithm.** A polynomial $f(x)$ is a *multiple* of $g(x)$ if $f(x) = h(x) \cdot g(x)$ for some polynomial $h(x)$. If $f(x)$ is not a multiple of $g(x)$, then there are polynomials $q(x)$ ("quotient") and $r(x)$ ("remainder") such that $f(x) = q(x) \cdot g(x) + r(x)$, where $r(x)$ has lower degree than $g(x)$.
- **Fundamental Theorem of Algebra.** A polynomial of degree n has exactly n complex roots, counted with multiplicity. In particular, it has at most n real roots. Furthermore, if the roots are $r_1, \ldots, r_n$, then $f(x) = c(x - r_1) \cdots (x - r_n)$ for some constant c.
- **Sum and Product of Roots.** If a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has roots (with repetition) $r_1, \ldots, r_n$, then
$$a_{n-1} = -(r_1 + \cdots + r_n); \qquad a_0 = (-1)^n r_1 \cdots r_n.$$
- **Repeated Roots.** A polynomial $f(x)$ is divisible by $(x - r)^k$ (i.e. the root r has multiplicity at least k) if and only if $f(r) = 0, f'(r) = 0, \ldots, f^{(k-1)}(r) = 0$.
- **Roots of Unity.** The roots of $x^n - 1$ are $1, e^{\frac{2\pi i}{n}}, e^{\frac{2\cdot 2\pi i}{n}}, \ldots, e^{\frac{(n-1)\cdot 2\pi i}{n}}$. These can also be written as $1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}$, where $\zeta_n := e^{\frac{2\pi i}{n}}$. The previous property implies that
$$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = 0.$$
- **Euler's Formula.** For real x, $e^{ix} = \cos(x) + i\sin(x)$.

Warm Up

---

1. Find the *rational factorization* of the following polynomials (this means all factors have rational coefficients):

(a) $x^3 - x^2 - x + 4$;

(b) $2x^4 - \frac{13}{2}x^2 + \frac{9}{2}$.

Hint: Try plugging in some simple values of x.

2. Show that there is a **unique** polynomial of the form $p(x) = x^2 + ax + b$ whose roots are also a and b; what are the values of a and b?

3. (a) The polynomial $x^4 + 2x^3 - 13x^2 - 14x + 24$ has roots $1, 3$, and -4. Without doing any polynomial division, find the fourth root.

   (b) Given that $f(x) = x^4 + 4x + 3$ has two roots r_1, r_2 such that $r_1 + r_2 = 2$, find the rational factorization of f.

4. Find a polynomial with **integer** coefficients and roots $-\frac{1}{2}, \frac{2}{3}$, and 2.

Main Problems

---

5. Find the rational factorization of $x^4 + x^2 + 1$.

6. Exactly **one** of the following polynomials is a multiple of $x^3 + 1$; determine which one:

$$x^{2017} + x^{1018} + x^{19} + x^{10} + 1 \qquad \text{or} \qquad x^{2017} + x^{1018} + x^{19} + x^{10} + x^9 + 1?$$

Hint: What happens if you plug in $x = -1$?

7. Find the largest n such that $n^3 + 121$ is a multiple of $n + 11$.

Hint: Write $n^3 + 121 = \left(n^3 + 1331\right) - 1210$ and use the division algorithm.

8. (a) Find b and c such that $x^2 + b$ and $x^3 + c$ are both multiples of the polynomial $x - 2$.

   (b) Show that for all $n \geq 1$ there is a unique integer a_n such that $x^n + a_n$ is a multiple of $x - 2$.

   Hint: There are at least two ways of approaching this problem: 1. Construct $a_n(x)$ inductively; 2. Use the polynomial division algorithm and consider the value at $x = 2$.

9. [**Gelca-Andreescu 304**] Let $p(x) = x^2 - 3x + 2$. Show that for any positive integer n there exist unique numbers a_n and b_n such that the polynomial $q_n(x) = x^n - a_n x - b_n$ is divisible by $p(x)$.

10. [VTRMC **1985 # 8**] Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where the coefficients a_i are real. Prove that $p(x)$ has at least one root in the interval $0 \leq x \leq 1$ if $a_0 + a_1/2 + \cdots + a_n/(n+1) = 0$.

11. [Putnam **1985 B1**] Let k be the smallest positive integer with the following property:

    There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

    has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

12. Let $f(x) := (x + 1/x)^4 - \left(x^4 + 1/x^4\right)$, where $x > 0$.

    (a) What is the limiting behavior of $f(x)$ as $x \to \infty$? As $x \to 0$?

    (b) Find the minimum value of $f(x)$. At which x is this value achieved?

13. [Putnam **1998 B1**] Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.