LSU Problem Solving Seminar - Fall 2017
Nov. 29: Putnam Review / Miscellaneous

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Putnam Mathematical Competition, Sat., Dec. 2
Lockett Hall 244, 8:30 A.M. – 5:00 P.M.

Test-taking tips:

- **Format.** The Exam is given in two 3-hour sessions of 6 problems each, with a lunch break from 12:00 – 2:00 P.M. The morning session's problems are labeled **A1** − **A6**, and the afternoon's **B1** − **B6**.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of 120. Typically there is very little partial credit given, and a submitted problem will receive **0, 1, 2, 9,** or **10** points.
- **A1/A2/B1.** In recent years these three problems have been the "easiest" part of the exam. More generally, the problems in each session are roughly ordered by difficulty. This is not an absolute rule, but you should expect that **A1** will have a relatively short solution, whereas **A6** will not. You should plan on spending at least 15 minutes each trying to make any progress on **A1, A2, B1** before moving on to the rest of the Exam.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending approximately one hour writing your solution. In light of the grading described above, it is better to solve one problem completely than several problems partially.

Main Problems

This week's practice sheet provides a detailed look at several problems from previous Putnam Exams. Each Exam problem is **preceded** by a related problem that illustrates some relevant concepts in a simpler context.

1. An *integer lattice path* from the origin to (m, n) is a sequence of alternating horizontal and vertical line segments (always moving in positive coordinate directions) such that H_1 begins at $(0, 0)$ and ends at some integer point $(a_1, 0)$, V_1 begins at $(a_1, 0)$ and ends at some integer point (a_1, b_1), and so on, until ending at (m, n).
   (a) How many distinct integer lattice paths are there that end at $(10, 10)$?
   (b) What proportion of these integer lattice paths pass through the point $(5, 5)$ along the way?

2. [Putnam **2011 A1**] Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0, 0), P_1, \ldots, P_n$ such that $n \geq 2$ and:
   - the directed line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc;
   - the lengths of these line segments are positive and strictly increasing.

How many of the points (x, y) with integer coordinates $0 \le x \le 2011, 0 \le y \le 2011$ *cannot* be the last point P_n of any growing spiral?

3. Recall that the *factorial* of a positive integer n is $n! := 1 \cdot 2 \cdots n$.

   (a) Prove that for all positive integers m, n, the quotient $\dfrac{(m+n)!}{m!n!}$ is also an integer.

   (b) Prove that if $m + n = p$ is prime, then the quotient $\dfrac{(m+n)!}{m!n!}$ is a multiple of p.

4. [Putnam **2009 B1**] Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

5. An *abelian group* $(G, \cdot)$ consists of a set of elements G and a binary operation $\cdot$ on G that satisfy the following axioms:

   - **Commutativity.** For all $a, b \in G$, $a \cdot b = b \cdot a$.
   - **Associativity.** For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   - **Identity.** There is an element $e \in G$ such that $a \cdot e = a$ for all $a \in G$.
   - **Inverses.** For each $a \in G$, there is an element $a^{-1} \in G$ such that $a \cdot a^{-1} = e$.

   (a) Prove that the identity element is unique; i.e., that if there is another element e' such that $a \cdot e' = a$ for all $a \in G$, then $e = e'$.

   (b) Prove that inverses are unique; i.e., that if $a \in G$ is given and b also satisfies $a \cdot b = e$, then $b = a^{-1}$.

6. [Putnam **2012 A2**] Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y). Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

7. Suppose that a closed box contains 1 ball of Color 1, 2 balls of Color 2, and so on, up to 10 balls of Color 10. What is the minimum number of balls that you must take in order to ensure that you have **all** of the balls of some color?

8. [Putnam **2010 B3**] There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

9. Prove that if n is a positive integer whose last digit is $1, 3, 7$, or 9, then there is some integer m such that $n \cdot m$ contains only the digit 7; i.e., $n \cdot m = 7777 \cdots 7$. For example, if $n = 21$, then $m = 37$ works, as $21 \cdot 37 = 777$.

10. [Putnam **2007 A4**] A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.