- Virginia Tech Mathematics Contest. Sat., Oct. 21. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 2. **Sign-up deadline: Oct. 6**.

LSU Problem Solving Seminar - Fall 2017

Sep. 13: The Pigeonhole Principle and other Invariants

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Useful facts and strategies:

- **Pigeonhole Principle.** If more than n objects are distributed among n sets, then some set contains multiple objects. **(Advanced Version.)** If more than nk objects are distributed among n sets, then one contains more than k objects.
- **Well-Ordering Property and Infinite Descent.** Every subset of natural numbers has a least element, or, equivalently, there is **no** infinite sequence of decreasing positive integers $n_1 > n_2 > n_3 > \cdots > 0$.

  This principle frequently applies to rational numbers as well; if r_i are rational numbers whose denominators are all **bounded**, then there is no infinite sequence $r_1 > r_2 > \cdots > 0$.
- **Invariants.** If you are asked about the final outcome of a procedure, try to find some property that remains the same at each step.
- **Monovariants.** If you are trying to show that a certain procedure ends in a **finite** number of steps, find a *monovariant:* a measurable quantity that is constantly increasing or decreasing.

Warm Up

1. There are 30 teams in Major League Baseball, 30 teams in the National Basketball Association, and 31 teams in the National Hockey League. Show that there are two students at LSU* who have the same favorite teams in all three leagues.

   This is still true even if you add a "None of the Above" option for each sport, as there may be students who do not follow Baseball, Basketball, and/or Hockey!

   * There are approximately 35,000 students at LSU.

2. Suppose that 10 odd, positive integers are written on the board. Two numbers a and b are chosen at random, erased, and replaced by their (non-negative) difference, $|a - b|$. If this procedure is continued until only one number remains, is it even or odd?

3. (a) The power goes out one morning before dawn while you are trying to get dressed. Your sock drawer contains 6 Purple socks, 7 Gold socks, and 10 White socks. How many must you grab in the dark in order to guarantee that you bring a matching pair with you?

   (b) Your closet contains 6 pairs of shoes, each consisting of a Left and Right foot. How many shoes must you grab in order to guarantee that you are able to walk without tripping (i.e., at least one Left and Right)?

4. A robot begins at the point $(0, 0)$ in the plane, and makes a sequence of moves. Each move must travel diagonally through a unit square. For example, the robot may begin by moving up and left, to $(-1, 1)$, and then up and right, to $(0, 2)$.

   (a) Is it possible for the robot to reach the point $(10, 2)$? If so, what is the minimum number of moves?

   (b) Is it possible for the robot to reach the point $(1, 2)$? If so, what is the minimum number of moves?

Main Problems

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5. A popular style of arithmetic puzzle is a *Cryptarithm*, where each letter is replaced by a digit so that the resulting expression is true. Each of the following puzzles has a **unique** solution; solve them systematically, without guessing:

(a)
$$\begin{array}{cccc} & \mathrm{V} & \mathrm{I} & \mathrm{I} \\ + & \mathrm{V} & \mathrm{I} & \mathrm{I} \\ \hline \mathrm{M} & \mathrm{I} & \mathrm{K} & \mathrm{E} \end{array}$$

(b)
$$\begin{array}{ccccccc} & \mathrm{C} & \mathrm{O} & \mathrm{O} & \mathrm{K} & \mathrm{I} & \mathrm{E} \\ & & & \mathrm{O} & \mathrm{R} & \mathrm{E} & \mathrm{O} \\ + & & & \mathrm{C} & \mathrm{O} & \mathrm{K} & \mathrm{E} \\ \hline & \mathrm{P} & \mathrm{U} & \mathrm{Z} & \mathrm{Z} & \mathrm{L} & \mathrm{E} \end{array}$$

6. A rectangular chocolate bar is segmented into small squares, with dimensions $m \times n$. The bar may be broken along any horizontal and vertical line between the squares, resulting in two smaller rectangles. Each of those smaller pieces can again be broken, continuing until all of the small squares have been separated from each other.

   (a) Describe a sequence of steps that breaks the bar completely into small squares. How many total breaks are in your method?

   (b) What is the minimum number of breaks required to separate the bar completely into small squares?

   (c) A family of rats have eaten several of the squares, leaving a single irregularly shaped piece that is composed of s connected squares. How many breaks are necessary to separate all of the squares in this piece?

7. (a) Prove that in any set of 7 integers, there are two whose difference is a multiple of 6.

   (b) Suppose that $n_1, n_2, \ldots, n_{k+1}$ are integers. Prove that there is some collection of consecutive n_i whose sum is a multiple of k. For example, in the sequence $2, -7, 2, 9, 2$ one sees that $-7 + 2 + 9 = 4$ is a multiple of 4.

   Hint: Consider the initial sums n_1, $n_1 + n_2$, $n_1 + n_2 + n_3, \ldots$.

8. (a) Prove that for any integer $k \geq 2$,

$$\frac{1}{k} + \frac{1}{k} < \frac{1}{k-1} + \frac{1}{k+1}.$$

(b) A group of 50 frogs are randomly placed on a line of 100 lily pads, with multiple frogs allowed on a pad. However, the frogs do not like to stay together, and so whenever there are two or more frogs sharing a pad, one will jump to the adjacent pad to the right, and one to the left. If multiple frogs are on the leftmost (1st) or rightmost (100th) pad, they will not jump any further.

Can this process continue indefinitely? Equivalently, is it ever possible for the frogs to return to an earlier configuation?

Hint: If the frogs are on pads $p_1, p_2, \ldots, p_{50}$, consider the quantity $\frac{1}{p_1} + \cdots + \frac{1}{p_{50}}$.

9. [VTRMC **1990 #3**] Let f be defined on the natural numbers as follows: $f(1) = 1$, and for $n > 1$, $f(n) = f(f(n-1)) + f(n - f(n-1))$. Find, with proof, a simple explicit expression for $f(n)$ which is valid for all $n = 1, 2, \ldots$.

10. (a) Consider a convex quadrilateral $ABCD$ in the plane. Prove that the sum of the two diagonal lengths is greater than the sum of any two **opposite** edges.

    (b) Suppose that a closed circuit through that points $A_1, A_2, \ldots, A_n$ is drawn in the plane. This is the sequence of line segments $\overline{A_1A_2}$, $\overline{A_2A_3}, \ldots$, $\overline{A_{n-1}A_n}$, $\overline{A_nA_1}$. Prove that if any line segments cross each other, then there is a **shorter** circuit through the points.

    Remark: If you rearrange edges, be sure that you still have a circuit, rather than creating separate loops!

11. [Putnam **1979 A4**] Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red, and the remaining n blue. Prove or disprove: there are n straight line segments, no two with a point in common, such that the endpoints of each segment are points of A with different colors.