- Virginia Tech Mathematics Contest. Sat., Oct. 21. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 2. **Sign-up deadline: Oct. 6**.

LSU Problem Solving Seminar - Fall 2017

Sep. 20: Calculus

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Useful facts and strategies:

- **Intermediate Value Theorem.** Suppose that $f(x)$ is a continuous function defined on the interval $[a,b]$, and r is a value in between $f(a)$ and $f(b)$, so that
$$f(a) < r < f(b) \quad \text{or} \quad f(a) > r > f(b).$$
Then there is some point c in the interior of the interval, $a < c < b$, such that $f(c) = r$.
In other words, a continuous function cannot "skip" any values.

- **Differentiability.** A function f is differentiable at a if the following limit exists:
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$
If so, this value is denoted by $f'(a)$.

- **Mean Value Theorem.** Suppose that $f(x)$ is differentiable on the interval $[a,b]$. Then there is a point $a < c < b$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
In other words, a differentiable function must achieve its **average** *slope at some point.*

- **Critical Points.** If $f(x)$ is a differentiable function on an interval $[a,b]$, then its maxima/minima must occur at the end points or the **critical points**, where are those x such that $f'(x) = 0$. The maxima/minima are classified by the negativity/positivity of $f''(x)$.
Something similar is true for multivariable functions; the maxima/minima of $f(x,y)$ also occur when $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are zero, but there is the additional possibility of a *saddle point*.

- **L'hospital's Rule.** Suppose $f(x)$ and $g(x)$ are differentiable. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an **indeterminate form** (i.e., $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

- **Continuity and Limits.** If g is a continuous function (or even just has a limit that exists at $f(a)$), then
$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x) \right).$$

- **Ordinary Differential Equations.** A differential equation of the shape
$$f'(x) = g(x) \cdot f(x)$$
has a solution $f(x) = e^{\int g(x)dx}$, where $\int g(x)dx$ denotes an antiderivative of g.

Warm Up

1. (a) Suppose that a and b are positive real numbers such that $ab = 64$. What is the minimum value of $a + b$?

   (b) Suppose that a, b, and c are positive real numbers such that $abc = 64$. What is the minimum value of $a + b + c$?

2. (a) Suppose that $f(x)$ is a differentiable function with roots at a and b, so $f(a) = f(b) = 0$. Prove that $f'(x) = 0$ for some $a < x < b$.

   Hint: Use the Mean Value Theorem.

   (b) Let $f(x) = x(x^2 - 1)(x^2 - 4)$. **Without** actually calculating the derivative, prove that $f'(x)$ has 4 real zeros.

3. Calculate the following limits.

   (a)
$$\lim_{x \to 0} \log\left(\frac{e^{2x} - 1}{x}\right),$$

   (b)
$$\lim_{x \to 0} \frac{\sqrt{\sin(4x^2)}}{x},$$

   (c)
$$\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x\right).$$

   Remark: This is an indefinite expression of the form $\infty - \infty$. It is often a good idea to use **conjugates** *to rewrite radical expressions; i.e. $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$.*

Main Problems

4. (a) Find the minimum value of $f(x) = x^2 + -2 + \dfrac{16}{x^2}$ over all positive x.

   (b) A rectangular area 400 square feet is to be fenced off. Find the dimensions that minimize the length of fencing used.

   (c) A reinforced shipping crate with volume 1500 cubic feet is to be built, where the bottom is doubly reinforced (this means that two layers of wood are used). What are the dimensions that minimize the amount of wood required?

   Remark: On a later practice sheet we will learn techniques for solving each of these **without** *calculus!*

5. Suppose that $f(x)$ is a continuous function such that $f'(x) = \dfrac{2x^4 - x^2 + \sin(x)}{x^4 + 2x + 2}$. Evaluate
$$\lim_{x \to \infty} \big(f(x + 2) - f(x)\big).$$

6. If $m \neq n$ are positive integers, find k such that the following limit exists and is nonzero:

$$\lim_{x \to 0} \frac{\cos(x)^m - \cos(x)^n}{x^k}.$$

Furthermore, find the value of the limit.

7. (a) Suppose that $x_0 > 1$ and $x_{n+1} = \frac{x_n^2+1}{2}$ for all $n \geq 1$. Prove that $\lim_{n\to\infty} x_n = \infty$.

   (b) [Gelca-Andreescu **386**] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Prove that f is constant.

   (c) [Gelca-Andreescu **383**] Does there exist a nonconstant function $f : (1, \infty) \to \mathbb{R}$ satisfying the relation $f(x) = f\left(\frac{x^2+1}{2}\right)$ for all $x > 1$ and such that $\lim_{x\to\infty} f(x)$ exists?

8. (a) Find the maximum value of $\ln\left(1 - x^2\right)$ for $-1 < x < 1$. What is the behavior near the endpoints, $x = \pm 1$?

   (b) [VTRMC **1998 #1**] Let $f(x, y) = \ln\left(1 - x^2 - y^2\right) - \dfrac{1}{(y - x)^2}$ with domain $D = \left\{(x, y) \mid x \neq y, x^2 + y^2 < 1\right\}$. Find the maximum value M of $f(x, y)$ over D. You have to show that $M \geq f(x, y)$ for every $(x, y) \in D$. Here $\ln(\cdot)$ is the natural logarithm function.

9. [Putnam **2011 B3**] Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0; if fg and f/g are differentiable at 0, must f be differentiable at 0?