LSU Problem Solving Seminar - Fall 2017
Oct. 4: Geometry and Trigonometry

Prof. Karl Mahlburg
Website: www.math.lsu.edu/$\sim$ mahlburg/teaching/Putnam.html

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Useful facts:

- **Triangle Inequality.** If a, b, c are the side lengths of a triangle, then $a < b + c$.
- **Pythagorean Theorem.** Suppose that ABC is a right triangle, with $\angle ABC = 90°$. If the (opposing) side lengths are $|\overline{AB}| = c$, $|\overline{AC}| = b$, $|\overline{BC}| = a$, then $b^2 = a^2 + c^2$.
- **Law of Cosines.** If a triangle has sides of lengths a, b, and c, and α is the angle opposite the side of length a, then
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
- **Law of Sines.** If β is the angle opposite b, and γ is the angle opposite c, then
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R},$$
where R is the radius of the circumscribed circle (which contains the vertices of the triangle).
- **Pythagorean Formula.** For all x,
$$\sin^2(x) + \cos^2(x) = 1,$$
- **Addition Formulas.** For all x and y,
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
- **Heron's Formula.** If a triangle has side lengths a, b, and c, then its area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s := \dfrac{a+b+c}{2}$ is the *semiperimeter.*
- **Prisms.** The volume of a prism of height h and base area A is $V = \dfrac{hA}{3}$.
- **Pick's Theorem.** Suppose that a polygon with integer vertices contains n integer points in its interior, and m integer points along its edges (including vertices). Then the area is $A = n + \dfrac{m}{2} - 1$.

Warm Up

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1. (a) Find the area of the region
$$\left\{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le x \le 1\right\}.$$
   (b) Find the area of the region
$$\left\{(x, y) \in \mathbb{R}^2 \mid 1 \le x \le 2,\ 0 \le y \le x\right\}.$$
   Try to find these areas geometrically, not with integrals!

2. (a) Suppose that ABC is a triangle with area 10 and perimeter 15. Construct a new figure by filling in all points that are within distance 1 of ABC. What is the area of this new figure?

   (b) How would your answer change if $ABCD$ were a convex quadrilateral with area 10 and perimeter 20?

3. (a) What is the area of the triangle with vertices $(0,0),(3,0)$, and $(0,4)$?

   (b) What is the area of the triangle with vertices $(0,0),(3,0)$, and $(10,4)$?

   (c) What is the area of the triangle with vertices $(0,0),(4,1)$, and $(2,5)$?

   Hint: Try to solve this by drawing a rectangle around the triangle (this is also a special case of Pick's Theorem)...

Main Problems

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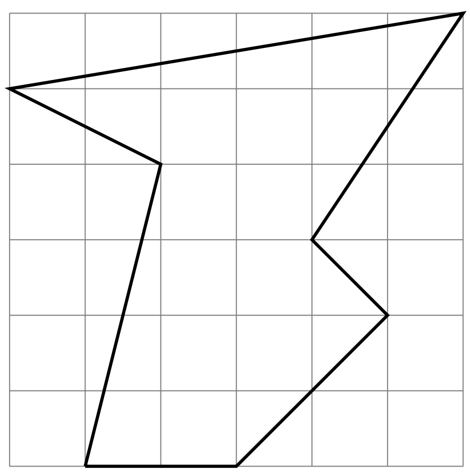
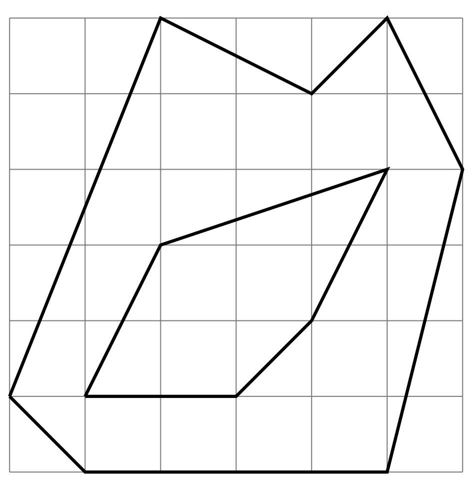
4. (a) Calculate the area of the following region:



   (b) Calculate the area **between** the two polygonal borders:



(c) Is it possible to draw an equilateral triangle in the plane such that all three vertices have integer coordinates?

Hint: What would Pick's Theorem imply if you had such a triangle?

5. (a) Calculate the volume of the region

$$\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; 0 \le z \le y \le x \le 1\right\}.$$

(b) Calculate the volume of the region

$$\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; 0 \le x \le 2,\ 0 \le z \le y \le x\right\}.$$

6. A *median* in a triangle connects a vertex to the midpoint of the opposing edge. The *centroid* of a triangle is the point at which the three medians meet.

(a) In a triangle ABC, denote the edge vectors by $\vec{u} := \overline{AB}$ and $\vec{v} := \overline{AC}$. Suppose that A is the origin, so that $\vec{u}$ and $\vec{v}$ correspond to the points B and C, respectively. Let O be the centroid of ABC. A common mistake is to believe that $O = \dfrac{\vec{u}}{2} + \dfrac{\vec{v}}{2}$. Show that in fact $\dfrac{\vec{u}}{2} + \dfrac{\vec{v}}{2}$ is the median of the edge $\overline{BC}$.

(b) Find the correct formula that expresses O in terms of $\vec{u}$ and $\vec{v}$.

(c) [Gelca-Andreescu **590**] Let M be a point in the plane of triangle ABC. Prove that the centroids of the triangles MAB, MAC, and MCB form a triangle similar to ABC.

7. A triangle ABC has (opposing) side lengths a, b, and c. The *circumcircle* is the unique circle containing the points A, B, and C.

Prove that the radius r of the circumcircle satisfies

$$r = \frac{abc}{4 \cdot \text{Area}(ABC)}.$$

Remark: As an additional exercise, explain why the circumcircle is **unique**.

8. Suppose that ABC is a triangle with opposing side lengths a, b, and c (i.e. $|\overline{AB}| = c$).

(a) Show that if $\angle ABC$ is **obtuse** (i.e., greater than $90°$), then $b^2 > a^2 + c^2$.

(b) What can you conclude about $\angle ABC$ if $b^2 < a^2 + c^2$?

9. [Putnam **2012 A1**] Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

10. Suppose that a region R in the plane has area A. For $m > 0$, let mR denote R "rescaled" by a factor of m; in other words,

$$mR := \{(mx, my) \mid (x, y) \in R\}.$$

What is the area of mR?

11. [Putnam **1994 A2**] Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.