LSU Problem Solving Seminar - Fall 2017
Oct. 11: Sequences and Series

Prof. Karl Mahlburg
Website: www.math.lsu.edu/∼mahlburg/teaching/Putnam.html

Useful facts:

- **Limit of a Sequence.** A sequence $\{a_n\}_{n=1}^{\infty}$ *converges to a limit* ℓ if for any $\varepsilon > 0$ there is an N such that $|a_n - \ell| < \varepsilon$ for all $n > N$.
- **Geometric Series.** If $|x| < 1$, then
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$
- **Ratio Test.** Let $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $L < 1$, then $\sum_{n\geq 1} a_n$ converges, and if $L > 1$, then the sum diverges. If $L = 1$, the test is inconclusive.
- **Monotone Convergence.** If $a_1 \leq a_2 \leq \ldots$ and all $a_n \leq B$ for some constant B, then $\lim_{n\to\infty} a_n$ exists.
- **Alternating Series.** If $a_1 \geq a_2 \geq \ldots$ and $\lim_{n\to\infty} a_n = 0$, then $a_1 - a_2 + a_3 - \ldots$ converges.
- **Integral Comparison.** If $f(x)$ is a decreasing function for $x \geq 0$, then $\sum_{n\geq 1} f(n) \leq \int_0^{\infty} f(x)dx$.
- **Linear Recurrences.** The *characteristic polynomial* associated to a (homogeneous) recurrence $a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n$ is $p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0$. If $p(x)$ has **distinct** roots $\lambda_1, \ldots, \lambda_k$, then the general solution to the recurrence is
$$a_n = b_1\lambda_1^n + \cdots + b_k\lambda_k^n,$$
where the constants are determined by k initial values. If there is a **repeated** root λ of order m, then the general solution has the term $(d_{k-1}n^{k-1} + \cdots + d_1 n + d_0)\lambda^n$.

Warm Up

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1. Evaluate the following sums:

   (a) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$.

   (b) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$.

2. (a) For any integer $n \geq 2$, evaluate the sum
$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n-2}{(n-1)!} + \frac{n-1}{n!}.$$
   What happens as $n \to \infty$?

   Hint: Calculate the first few cases and look for a pattern.

(b) Evaluate the sum

$$1 + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!} + \cdots .$$

Main Problems

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3. Consider the set of integers whose only prime factors are 2 or 5. Evaluate the sum of their reciprocals:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \cdots .$$

4. At the unfriendliest bar in town, no two people will ever sit on adjacent stools.

   (a) If there are 4 stools, write down all possible seating arrangements (including the case with no people). For example, it is permissible to have seats 1 and 4 occupied, but it is not allowed to have 1, 3, and 4.

   (b) Business is booming, so the seating is expanded to 6 stools. How many possible seating arrangements are there?

   (c) Now suppose that there are n stools, where n is a nonnegative integer, and let b_n be the number of possible seating arrangements. Show that for $n \geq 2$ these numbers satisfy the recurrence

$$b_n = b_{n-1} + b_{n-2}.$$

5. The *Riemann zeta function* is defined for $s > 1$ by

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots = \sum_{n \geq 1} \frac{1}{n^s}.$$

   (a) If you try to plug in $s = 1$, the resulting expression is known as the *Harmonic series.* Prove that this diverges, i.e., that

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \to \infty.$$

   (b) Prove that for $s > 1$, there is a simple bound $\zeta(s) < 1 + \dfrac{1}{s-1}$.

   (c) Evaluate the sum

$$\sum_{k=2}^{\infty} \big(\zeta(k) - 1\big).$$

   The answer is **extremely** simple.

6. (a) Prove that $\lim\limits_{x \to \infty} x \sin\left(\dfrac{1}{x}\right) = 1$.

   (b) Prove that there are infinitely many integers n such that $\sin(n) > \dfrac{1}{2}$.

7. [Gelca-Andreescu **357**] Does the series $\displaystyle\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2+1}\right)$ converge?

8. [VTRMC **1990 #5**] Determine all real values of p for which the following series converge.

(a) $\displaystyle\sum_{n\geq 1}\left(\sin\left(\frac{1}{n}\right)\right)^p$,

(b) $\displaystyle\sum_{n\geq 1}|\sin(n)|^p$.

9. [Putnam **2013 B1**] For positive integers n, let the number $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

10. [Putnam **1985 A3**] Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}, j = 0, 1, 2, \ldots$ by the condition

$$\begin{aligned} a_m(0) &= \frac{d}{2^m}, \\ a_m(j+1) &= (a_m(j))^2 + 2a_m(j), \qquad j \geq 0. \end{aligned}$$

Evaluate $\lim\limits_{n\to\infty} a_n(n)$.