- Virginia Tech Mathematics Contest. Sat., Oct. 21, Lockett 232 / 244.

  8:30–8:45 A.M. Breakfast.

  8:45–9:00 A.M. Registration Form.

  9:00–11:30 A.M. Exam.

LSU Problem Solving Seminar - Fall 2017
Oct. 18: Integration

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Useful facts:

- **Partial Fractions.** If $f(x)$ is a polynomial whose degree is less than n, then there are constants $a_1, \dots, a_n$ such that

$$\frac{f(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{x-r_1} + \cdots + \frac{a_n}{x-r_n}.$$

  (Here the roots r_i must be distinct – there is a more complicated version for repeated roots.)

- **Fundamental Theorem(s) of Calculus.** Suppose that $f(x)$ is a continuous function.
  - If $F(x)$ is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$.
  - Define $g(x) := \int_a^x f(t)dt$. Then $g'(x) = f(x)$.

- **Integration By Parts.** Suppose that f and g are differentiable. Then

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$

- **Substitution.**

$$\int_{x=a}^{b} f(u(x))u'(x)dx = \int_{u=u(a)}^{u(b)} f(u)du.$$

- **Symmetries and Substitution.** Remember, integration problems on Mathematics Contests are meant to have solutions! A complicated integral often has a hidden symmetry or substitution that makes it much easier to evaluate. For example, if $f(x)$ is an *odd* function, then $\int_{-a}^{a} f(x)\, dx = 0$.

Warm Up

1. Calculate the following antiderivatives (indefinite integrals):

   (a) $\int \frac{1}{2x+3}\, dx$,

(b) $\int x^2 e^{x^3}\, dx$

(c) $\int \frac{x^2}{x^2+4}\, dx,$
Hint: Try the substitution $x = \tan u$.

2. Evaluate the following integrals with as **little** computation as possible:

(a) $\int_0^4 \sqrt{4x - x^2}\, dx.$
Hint: Complete the square and think geometrically.

(b) $\int_{-1}^{1} \frac{\sin(x)}{e^{x^2} + x^2}\, dx.$

3. Find a function f such that

$$\int_0^1 f(x)dx = 1 \qquad \text{and} \qquad \int_0^1 x^2 f(x)dx = 1.$$

Main Problems

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4. Evaluate the following integral – the answer is **very** simple!

$$\int_0^1 \frac{3^{x^3}}{2} + \sqrt[3]{\log_3(2x+1)}\, dx = ?$$

Hint: Write the integrand in the form $f(x) + f^{-1}(x)$*, and then draw a picture. . . .*

5. The *dilogarithm* function is defined by

$$\log_2(z) := -\int_0^z \frac{\log(1-x)}{x} dx.$$

(a) Prove that the Taylor expansion (around 0) is

$$\log_2(z) = z + \frac{z^2}{4} + \frac{z^3}{9} + \cdots = \sum_{k \geq 1} \frac{z^k}{k^2}.$$

Remark: The name dilogarithm is motivated by the squares in the denominator. The value at $z = 1$ *is famously* $\log_2(1) = \zeta(2) = \frac{\pi^2}{6}$.

(b) Find a change of variables to show the following alternative integral representation at $z = 1$:

$$\int_0^\infty \frac{x}{e^x - 1} dx = \log_2(1).$$

6. When integrating rational functions, the *logarithmic derivative* formula is often helpful:
$\int \frac{f'(x)}{f(x)}\, dx = \ln(f(x)) + C.$

(a) Find the antiderivative

$$\int \frac{(x+1)^2}{x^2+1}\,dx.$$

(b) [Gelca-Andreescu **445**] Compute

$$\int \frac{x+\sin x-\cos x-1}{x+e^x+\sin x}\,dx.$$

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7. (a) Suppose that $f(x)$ is a function such that $f(x) \neq -f(-x)$ at any point in $[-a,a]$. Evaluate the integral

$$I := \int_{-a}^{a} \frac{f(x)}{(f(x)+f(-x))}\,dx.$$

Hint: Write $I+I$, and in the second term make the substitution $x=-u$.

(b) Evaluate

$$\int_0^4 \frac{\sqrt{1+x}}{\sqrt{1+x}+\sqrt{5-x}}\,dx.$$

Hint: Try to rewrite the integral so that it matches the general form from (a).

8. [VTRMC **2012 #1**] Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos^4 x+\sin x\cos^3 x+\sin^2 x\cos^2 x+\sin^3 x\cos x}{\sin^4 x+\cos^4 x+2\sin x\cos^3 x+2\sin^2 x\cos^2 x+2\sin^3 x\cos x}\,dx.$$

9. For $r \geq 0$, define the integral

$$I_r := \int_0^{\frac{\pi}{2}} x^r \sin x\,dx.$$

(a) Use integration by parts (twice!) to show that

$$I_r = \frac{1}{r+1}\left(\frac{\pi}{2}\right)^{r+1} - \frac{1}{(r+1)(r+2)} I_{r+2}.$$

(b) Conclude that

$$\begin{aligned} I_r &= \frac{1}{r+1}\left(\frac{\pi}{2}\right)^{r+1} - \frac{1}{(r+1)(r+2)(r+3)}\left(\frac{\pi}{2}\right)^{r+3} + \dots \\ &= r!\left(\frac{1}{(r+1)!}\left(\frac{\pi}{2}\right)^{r+1} - \frac{1}{(r+3)!}\left(\frac{\pi}{2}\right)^{r+3} + \frac{1}{(r+5)!}\left(\frac{\pi}{2}\right)^{r+5} - \dots\right). \end{aligned}$$

Is this a meaningful expression; i.e., does it converge?

(c) Finally, prove that the first term is dominant for large r. In particular, that

$$\lim_{r\to\infty} \frac{I_r}{\frac{1}{r+1}\left(\frac{\pi}{2}\right)^{r+1}} = 1.$$

10. [Putnam **2011 A3**] Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int\limits_0^{\pi/2} x^r \sin x \, dx}{\int\limits_0^{\pi/2} x^r \cos x \, dx} = L.$$

11. [Putnam **1995 A2**] For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right)$$

converge?