LSU Problem Solving Seminar - Fall 2017
Oct. 25: Virginia Tech Math Contest Review

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This week's practice sheet provides a detailed look at several of the problems from last weekend's 2017 Virginia Tech Regional Math Contest. Each contest problem is **preceded** by a related problem that illustrates some relevant techniques in a simpler context.

1. (a) Consider a quarter-circle of radius r in the first quadrant, which has the equation $y = f(x) = \sqrt{r^2 - x^2}$ for $0 \le x \le r$. Prove that any line intersects the circle in at most two points.

   (b) Suppose that $f(x)$ is a decreasing function on $[0, \infty)$ that is also *concave down* – this means that $f''(x) < 0$ for all x. Prove that any line intersects the graph of $f(x)$ in at most two points.

   Remark: Try to find both a geometric argument and a calculus argument!

2. [VTRMC **2017 # 1**] Determine the number of real solutions to the equation $\sqrt{2 - x^2} = \sqrt[3]{3 - x^3}$.

   Hint: Find the intersections of each curve with the lines $y = c - x$.

3. Recall the following trigonometric identities and derivatives:

$$\tan^2(u) + 1 = \sec^2(u),$$
$$\frac{d}{du}\tan(u) = \sec^2(u), \qquad \frac{d}{du}\sec(u) = \sec(u)\tan(u).$$

   (a) Try to prove the identities above!

   (b) Use the substitution $x = \tan(u)$ to evaluate the integral $\int_0^\infty \frac{1}{x^2+1} dx$.

   (c) Calculate the antiderivative $\int \sec(x) dx$ by first writing $\sec(x) = \frac{\sec(x)(\sec(x) + \tan(x))}{\sec(x) + \tan(x)}$, and then making the substitution $u = \sec(x) + \tan(x)$.

4. [VTRMC **2017 # 2**] Evaluate $\int_0^a \frac{dx}{1 + \cos x + \sin x}$ for $-\pi/2 < a < \pi$. Use your answer to show that $\int_0^{\pi/2} \frac{dx}{1 + \cos x + \sin x} = \ln 2$.

   Hint: Try to rewrite the integrand in terms of $\sec(x)$ *and* $\tan(x)$.

5. Suppose that a triangle ABC has angles $\alpha = \angle BAC, \beta = \angle ABC$, and $\gamma = \angle ACB$, and side lengths $a = |\overline{BC}|, b = |\overline{BC}|$, and $c = |\overline{AB}|$.

   (a) The Law of Sines states that $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$, and the Law of Cosines states that $a^2 = b^2 + c^2 - 2bc\cos\alpha$. Try to prove these!

   (b) The Double-Angle formula for the sine function states that $\sin(2x) = 2\sin x\cos x$. Try to prove this too!

6. [VTRMC **2017 # 3**] Let ABC be a triangle and let P be a point in its interior. Suppose $\angle BAP = 10°, \angle ABP = 20°, \angle PCA = 30°$, and $\angle PAC = 40°$. Find $\angle PBC$.

7. In this problem you will explore the use of *barycentric* coordinates in a triangle. If P is a point in a triangle ABC, the distance from P to the edge $\overline{AB}$ is defined to be the length of the perpendicular line segment joining P to (the line extending) $\overline{AB}$, and is denoted by $d(P, \overline{AB})$.

   (a) Suppose that ABC is an equilateral triangle of side length ℓ. Prove that for every point P in the triangle,
   $$d(P, \overline{AB}) + d(P, \overline{AC}) + d(P, \overline{BC}) = \frac{\sqrt{3}}{2} \cdot \ell.$$

   (b) Consider the plane (in $\mathbb{R}^3$) that intercepts the axes at $(x_0, 0, 0), (0, y_0, 0)$, and $(0, 0, z_0)$ (with $x_0, y_0, z_0 > 0$). Find the formula for this plane in the form
   $$dx + ey + fz = k,$$
   where d, e, f, and k are constants that you need to determine.

   (c) The intersection of this plane with the first octant (i.e., where $x, y, z \geq 0$) is a triangle. The *barycentric* coordinates for a point on this triangle are $\left(\frac{x}{x_0}, \frac{y}{y_0}, \frac{z}{z_0}\right)$.

   (d) Suppose that a triangle has side lengths a, b, and c. Show that it can be embedded in the first octant of three-dimensional Cartesian space as in part (b); in particular, show how x_0, y_0, and z_0 are determined by a, b, and c.

8. [VTRMC **2017 # 4**] Let P be an interior point of a triangle of area T. Through the point P, draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let a, b, and c be the areas of the three triangles. Prove that $\sqrt{T} = \sqrt{a} + \sqrt{b} + \sqrt{c}$.

9. (a) Suppose that x and y are positive integers such that xy is a perfect square. Show that there are positive integers a, b, and s such that $x = a^2 s, y = b^2 s$, and a and b have no common factors (i.e., are "relatively prime").

   (b) For positive real numbers x and y, the *arithmetic, geometric,* and *harmonic* means are defined, respectively, as
   $$A(x, y) := \frac{x + y}{2}, \; G(x, y) := \sqrt{xy}, \; H(x, y) := \frac{2xy}{x + y}.$$
   Prove that $H(x, y) \leq G(x, y) \leq A(x, y)$, with equality if and only if $x = y$.

10. [VTRMC **2017 # 5**] Let $f(x, y) = \dfrac{x + y}{2}$, $g(x, y) = \sqrt{xy}$, $h(x, y) = \dfrac{2xy}{x + y}$, and let
    $$S = \big\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \neq b \text{ and } f(a, b), g(a, b), h(a, b) \in \mathbb{N}\big\},$$
    where $\mathbb{N}$ denotes the positive integers. Find the minimum of f over S.

    Hint: The minimum value occurs in **two** *distinct ways. . . .*

11. The *polynomial division algorithm* implies that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is a polynomial with integer coefficients, then for any $k \in \mathbb{Z}$ there is some polynomial $g(x) \in \mathbb{Z}[x]$ and constant $b \in \mathbb{Z}$ such that

$$f(x) = (x - k) \cdot g(x) + b.$$

   (a) Show that in fact the constant in the polynomial division algorithm is given by $b = f(k)$.

   (b) Suppose that $f(x) \in \mathbb{Z}[x]$ is a polynomial with integer coefficients such that $f(1) = 2$. Determine all possible $n \in \mathbb{Z}$ such that $f(n) = 21$.

12. [VTRMC **2017 # 6**] Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1, f(4) = 2$, and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for n.