MATH 7230 Homework 1 - Spring 2017

Due Thursday, Jan. 26 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

1. In this problem you will describe the conjugacy classes of the symmetric group S_n. Recall that two elements g, h in a group G are *conjugates* if $g = aha^{-1}$ for some $a \in G$.
   (a) Suppose that $(a_1\ a_2\ \cdots\ a_k) \in S_n$; this is a k-cycle. Prove that for any $\sigma \in S_n$,
   $$\sigma(a_1\ a_2\ \cdots\ a_k)\sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \cdots\ \sigma(a_k)) .$$
   (b) Show that any two k-cycles are conjugates.
   (c) Prove that conjugacy classes in S_n are determined by partitions $\lambda \vdash n$. In particular, if $\sigma \in S_n$ has the disjoint cycle factorization $\sigma = \sigma_1\sigma_2\cdots\sigma_r$, where σ_i is a λ_i-cycle, then the conjugacy class of σ consists of all permutations that factor into disjoint cycles of lengths $\lambda_1, \ldots, \lambda_r$.
2. Provide a bijective proof of Corollary 1.3. In particular, for $d \geq 1$ let $p_{d+1}(n)$ be the number of partitions of n into parts that are **not** multiples of $d+1$, and let $Q_{\leq d}(n)$ denote the number of partitions of n in which each part occurs at most d times. Prove that
   $$p_{d+1}(n) = Q_{\leq d}(n).$$
3. Andrews 1.3. This is easiest using generating functions. Can you also find a bijective proof?
4. Andrews 1.9. Draw Ferrers diagrams and consider the conjugates.

In the following several problems you will explore an enumeration function that is closely related to partitions, although unlike partitions, this case does turn out to have a simple closed formula. An integer *composition* of n is a sequence of positive integers $\mu_1, \ldots, \mu_r$ that sum to n. Here there is no restriction on the ordering of the parts, and each distinct sequence is counted. For example, the compositions of 4 are:

$$4,\ 3+1,\ 1+3,\ 2+2,\ 2+1+1,\ 1+2+1,\ 1+1+2,\ 1+1+1+1.$$

The *composition function* $c(n)$ counts the number of compositions of n; the above example shows $c(4) = 8$.

5. (a) Find the value of $c(n)$ for $n = 1, 2$ and 3. By convention, we again set $c(0) = 1$. Conjecture a formula for $c(n)$.

(b) Prove the recurrence formula (valid for $n \geq 1$)

$$c(n) = \sum_{j=1}^{n} c(n-j).$$

Hint: The j-th term corresponds to compositions of n whose first part is j.

(c) Use the previous part to give an inductive proof of the formula for $c(n)$.

6. Define the generating function for compositions by $C(q) := \sum_{n \geq 0} c(n) q^n$ (here q is a formal variable, and $C(q)$ is in the power series ring $\mathbb{R}[[q]]$).

   (a) Let $c_j(n)$ be the number of compositions of n with **exactly** j parts. Explain why

$$\sum_{n \geq 0} c_j(n) q^n = \frac{q^j}{(1-q)^j}.$$

   (b) Next argue that $c(n) = \sum_{j \geq 1} c_j(n)$. Conclude that

$$C(q) = 1 + \frac{q}{1-q} + \frac{q^2}{(1-q)^2} + \cdots .$$

   (c) Evaluate the geometric series to find a closed-form expression for $C(q)$. How does this relate to Problem 5?

7. This series of problems ends with an alternative approach to enumerating compositions.

   (a) Let $\mathcal{C}_n$ denote the set of compositions of n. For $n \geq 1$, define two maps $\beta, \gamma : \mathcal{C}_n \mapsto \mathcal{C}_{n+1}$ by the following operations:

$$\beta(\mu_1, \mu_2, \ldots, \mu_r) := (1, \mu_1, \mu_2, \ldots, \mu_r);$$
$$\gamma(\mu_1, \mu_2, \ldots, \mu_r) := (\mu_1 + 1, \mu_2, \ldots, \mu_r).$$

   Prove that β and γ disjointly produce all compositions of $n+1$; i.e.,

$$\beta(\mathcal{C}_n) \cup \gamma(\mathcal{C}_n) = \mathcal{C}_{n+1} \qquad \text{and} \qquad \beta(\mathcal{C}_n) \cap \gamma(\mathcal{C}_n) = \emptyset.$$

   Conclude that $c(n+1) = 2c(n)$, and compare to Problem 5.

   (b) The same idea can be encoded using generating functions. Explain the details of the following recurrence:

$$C(q) = \underbrace{1}_{\text{Empty composition}} + \underbrace{q \cdot C(q)}_{\text{If } \mu_1 = 1\text{, remove it}} + \underbrace{q \cdot (C(q) - 1)}_{\substack{\text{If } \mu_1 > 1\text{, descrease by 1} \\ \text{Resulting composition is nonempty}}} .$$

   Compare to Problem 6.