MATH 7230 Homework 2 - Spring 2017

Due Thursday, Feb. 2 at 10:30

www.math.lsu.edu/∼mahlburg/

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

1. The *q-factorial* (we will explore the reason for this name later) is defined by

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right).$$

   (a) Prove the following identity inductively:

$$(-q;q)_n = 1 + \sum_{j=1}^{n} q^j \cdot (-q;q)_{j-1}.$$

   (b) Now prove the identity combinatorially. The left-hand side generates partitions into distinct parts, all of which are at most n. For the right-hand side, explain why the j-th term generates those partitions into distinct parts whose largest part is j.

In the following problems you will prove some basic facts about the ring of formal power series. See Niven's paper (available on course webpage) for more details. If R is a ring (at minimum, an integral domain), then the *formal power series ring on* R is

$$P_R := \left\{\alpha = [a_0, a_1, a_2, \dots] \mid a_n \in R \text{ for all } n \geq 0\right\},$$

with ring operations (here $\beta = [b_0, b_1, \dots]$)

$$\alpha + \beta := [a_n + b_n]_{n=0}^{\infty};$$

$$\alpha \cdot \beta := [c_n]_{n=0}^{\infty}, \quad \text{with } c_n := \sum_{i=0}^{n} a_i b_{n-i}.$$

2. (a) Prove that P_R is a commutative ring with unity. The additive identity is $0_P := [0,0,\dots]$, and the multiplicative identity is $1_P := [1,0,\dots]$.

   (b) Furthermore, prove that P_R is an integral domain (again, assuming that R is).

   (c) Show that $[1,-1,0,0,\dots] \cdot [1,1,1,\dots] = 1_P$.

   Remark: For this and the next part, you may find it easier to use the notation from Problem 3.

(d) Prove that α has a multiplicative inverse α^{-1} (such that $\alpha \cdot \alpha^{-1} = 1_P$) if and only if a_0 is invertible in R (if R is a field, this just means $a_0 \neq 0$).

3. P_R is related to the more typical notation for power series by setting $q := [0, 1, 0, 0, \dots]$ and $q^j := [0, \dots, 0, 1, 0, \dots]$, where $a_j = 1$. Then P_R is isomorphic to $R[[q]]$ under the identification

$$\alpha \leftrightarrow a_0 + a_1 q + a_2 q^2 + \dots.$$

(a) If R is a field with characteristic 0 (i.e. $\mathbb{R}$), then it is a fact that roots of power series exist. For example, if

$$\alpha := 1 + 2q + 3q^2 + 4q^3 + \dots,$$

find its square root β, which satisfies $\beta \cdot \beta = \alpha$.

(b) Now find the square root of β.

(c) The generalized Binomial Theorem states that for any complex γ,

$$(1+q)^\gamma = \sum_{n \geq 0} \frac{\gamma \cdot (\gamma - 1) \cdots (\gamma - n + 1)}{n!} q^n.$$

This is frequently proven as an analytic result via Taylor series. Compare to your answers from parts (a) and (b).

In these next problems you will consider the analytic convergence of some of the infinite products and sums that arise in the study of partitions. If $\mathbf{a} := (a_n)_{n \geq 1}$ is a sequence of complex numbers, let

$$P(\mathbf{a}) := \prod_{n \geq 1} (1 + a_n) \text{ and } S(\mathbf{a}) := \sum_{n \geq 1} a_n.$$

4. First, consider the case that all a_n are real.

(a) Prove that $1 + x \leq e^x$ for all real x.

(b) Suppose that $a_n \geq 0$ for all n. Prove that

$$S(\mathbf{a}) \leq P(\mathbf{a}) \leq e^{S(\mathbf{a})}.$$

Thus $S(\mathbf{a})$ converges if and only if $P(\mathbf{a})$ does!

(c) Prove that if $a_n \in (-1, \infty)$ for all n, then the convergence of $S(\mathbf{a})$ implies the boundedness of the partial products of $P(\mathbf{a})$, which are $\prod_{n=1}^{N} (1 + a_n)$.

Remark: In general, the convergence of $S(\mathbf{a})$ and $P(\mathbf{a})$ are unrelated. For example, consider $\left(1 + \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{2}} + \frac{1}{2}\right)\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{3}} + \frac{1}{3}\right)\cdots$, *so*

$$a_{2n-1} := 1 + \frac{1}{\sqrt{n}}; \quad \textit{and} \quad a_{2n} := 1 - \frac{1}{\sqrt{n}} + \frac{1}{n}.$$

Then $P(\mathbf{a})$ converges, but $S(\mathbf{a})$ diverges!

(d) Why is the range in part (c) restricted to $a_n > -1$?

5. The infinite product denotes a **limit**, i.e.

$$P(\mathbf{a}) := \lim_{N\to\infty} \prod_{n=1}^{N} (1 + a_n).$$

(a) Use Problem 4 to prove that if $a_n \in \mathbb{C}$ satisfy $|a_n| < 1$ for all n, and $S(\mathbf{a})$ converges absolutely (i.e., $\sum |a_n|$ converges), then $P(\mathbf{a})$ converges.

(b) Conclude that $\prod_{n\geq 1} (1 - q^n)$ converges for $|q| < 1$.

(c) Conclude further that the product in part (b) is non-zero (so that $P(q)$ converges as well).

6. Prove that if $|q| < 1$, the following sums converge absolutely:

(a) $\displaystyle\sum_{n\geq 1} \frac{q^n}{\prod_{j=1}^{n}(1 - q^j)}$;

(b) $\displaystyle\sum_{n\geq 1} q^n \prod_{j=1}^{n-1} (1 + q^j)$.

Remark: This shows that it makes sense analytically to take the limit as $n \to \infty$ in Problem 1.