MATH 7230 Homework 3 - Spring 2017

Due Thursday, Feb. 16 at 10:30

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You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

The first set of problems illustrate some additional combinatorial settings where partitions naturally occur.

1. (a) For nonnegative integers $n_1, n_2, \ldots, n_r$, the *multinomial coefficient* is defined by

$$\binom{n_1 + n_2 + \cdots + n_r}{n_1, n_2, \ldots, n_r} := \frac{(n_1 + n_2 + \cdots + n_r)!}{n_1! n_2! \cdots n_r!}.$$

This enumerates the number of distinct words formed with the symbols $a_1, \ldots, a_r$, where there are n_i (indistinguishable) copies of a_i. For example, $\binom{4}{2,1,1} = 12$, and there are indeed 12 words formed from the symbols a, a, b, c :

aabc, *aacb*, *abac*, *abca*, *acab*, *acba*, *baac*, *baca*, *bcaa*, *caab*, *caba*, *cbaa*.

Explain why this is equivalent to choosing tiered rankings for $n = n_1 + \cdots + n_r$ people, where n_1 people are given first ranking, n_2 people are given second ranking, and in general, n_i are given i-th ranking.

(b) Prove that multinomial coefficients arise from the following product:

$$(1 + x_1 + \ldots x_r)^n = \sum_{\substack{n_i \geq 0 \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}.$$

(c) Both parts above immediately show that the multinomial coefficients are integers; i.e., that $(n_1 + \cdots + n_r)!$ is divisible by $n_1! \cdots n_r!$ for all $n_i \geq 0$. In fact, more is true: if $(n_1, n_2, \ldots, n_r) = 1^{f_1} 2^{f_2} \cdots n^{f_n}$ in partition frequency notation, where $n = n_1 + \cdots + n_r$, then

$$\frac{(n_1 + n_2 + \cdots + n_r)!}{n_1! \cdots n_r! f_1! \cdots f_n!}$$

is an integer. In particular, it counts the number of distinct *set partitions* of $[n] := \{1, 2, \ldots, n\}$, i.e., a disjoint union $[n] = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r$, where $|S_i| = n_i$.

Remark: Even more remarkably, fixing n *and summing over all partitions* $\lambda = (\lambda_1, \ldots, \lambda_r) = 1^{f_1} \cdots n^{f_n}$ *gives the total number of set partitions into* r *subsets, which are enumerated by the second Stirling numbers,*

$$S(n, r) = \sum_{\substack{\lambda \vdash n, \\ \ell(\lambda) = r}} \frac{n!}{\lambda_1! \cdots \lambda_r! f_1! \cdots f_n!}.$$

It is known from entirely different methods that $S(n, r) = \frac{1}{r!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^n$.

2. (a) Consider all integers from 0 to $10^k - 1$, writing each one with k digits: $N = d_{k-1} \cdots d_1 d_0$, $d_i \in \{0, 1, \dots 9\}$. Each such N has a partition $\lambda(N) \vdash k$ associated to its it digit multiplicities, which are the number of occurrences of each digit. For example, $N = 7337$ has $\lambda(N) = (2, 2)$ since there are two distinct digits, each occurring twice, whereas for $N = 0506$, $\lambda(N) = (2, 1, 1)$.

   Now count the N in reverse, grouping integers by the associated partition. In particular, suppose that $\lambda \vdash k$, with parts $(\lambda_1, \dots, \lambda_\ell)$ and frequencies $\lambda = 1^{f_1} \cdots k^{f_k}$ (note that $\ell \leq 10$ and $f_1 + \cdots + f_k = \ell$). Show that the number of N associated to λ is

$$\binom{10}{f_1, f_2, \dots, f_k, 10 - \ell} \binom{k}{\lambda_1, \dots, \lambda_\ell}.$$

   For example, if $k = 4$ and $\lambda = (3, 1)$, we are counting 4-digit integers with three copies of one digit and one copy of another (e.g. 2272 or 8881). The claimed formula is explained by

$$\binom{10}{1, 1, 8} \binom{4}{3, 1} = \underbrace{10}_{\substack{\text{Pick digit to} \\ \text{occur 3 times}}} \cdot \underbrace{9}_{\substack{\text{Pick digit to} \\ \text{occur 1 time}}} \cdot \underbrace{4}_{\substack{\text{Pick position for} \\ \text{single digit}}} = 360.$$

   (b) Plug in for all partitions $\lambda \vdash 4$, and make sure that your computations sum up to a total of 10,000.

3. In this problem you will prove Faa di Bruno's formula, which calculates higher derivatives for composite functions (i.e. repeated differentiation of the Chain Rule). The statement is that for $n \geq 0$, and f, g sufficiently differentiable functions,

$$\frac{d^n}{dx^n} f\big(g(x)\big) = \sum_{\substack{\text{For } 1 \leq i \leq n, f_i \geq 0: \\ 1 \cdot f_1 + 2 \cdot f_2 + \cdots + n \cdot f_n = n}} \frac{n!}{1!^{f_1} 2!^{f_2} \cdots n!^{f_n} f_1! \cdots f_n!} f^{(f_1 + \cdots + f_n)}\big(g(x)\big) \prod_{j=1}^{n} \left(g^{(j)}(x)\right)^{f_j}.$$

   Note that the initial part of the formula can be rewritten more explicitly in terms of partitions:

$$\sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n, \\ \lambda = 1^{f_1} 2^{f_2} \cdots n^{f_n}}} \frac{n!}{\lambda_1! \cdots \lambda_\ell! f_1! \cdots f_n!}$$

   (a) Verify the formula for $n = 1, 2, 3$.

   (b) Prove the general case. Try to induct on n.

In problems 4–5 you will continue working in the ring of formal power series with complex coefficients. See Niven's paper (available on course webpage) for more details.

4. In $\mathbb{C}[[q]]$, a series $\sum_{n \geq 0} a_n$ is said to be *summable* if for any $N \in \mathbb{N}$ there is an M such that for $n > M$, a_n is $O(q^N)$, which means that $a_n = q^N \cdot b_n$ for some $b_n \in \mathbb{C}[[q]]$. In other words, the terms a_n "eventually" involve only larger and larger powers of q.

(a) What follows is an incorrect calculation. Explain what is wrong and how it can be repaired:

$$\begin{aligned}\sum_{j\geq 0}\frac{q^j}{(q;q)_j} &= \sum_{j\geq 0}\frac{(q^j-1)+1}{(q;q)_j}\\ &= \sum_{j\geq 1}\frac{-1}{(q;q)_{j-1}}+\sum_{j\geq 0}\frac{1}{(q;q)_j}\\ &= -\sum_{j\geq 0}\frac{1}{(q;q)_j}+\sum_{j\geq 0}\frac{1}{(q;q)_j}=0.\end{aligned}$$

(b) A formal power series can be summable even if it is poorly behaved analytically. For example, let $F(x) := \sum_{n\geq 0} n!x^n$ be the generating function for factorials; note that this diverges for **all** complex numbers $x \neq 0$! However, as a formal power series, $F(x)$ even has a multiplicative inverse.

Let $C_1 := 1$, and for $n \geq 2$,

$$C_n := n! - \sum_{m=1}^{n-1} C_m \cdot (n-m)!.$$

Prove that $F(x)^{-1} = 1 - \sum_{n\geq 1} C_n$.

Remark: C_n enumerates the number of connected *permutations of n, which are those that are not a disjoint product of the form $\sigma_m\tau$, where σ_m is a permutation on $1,\ldots,m$.*

5. This problem uses formal power series to prove a relation between elementary number theory and partitions. You may use the fact that basic functions/operations such as logarithms and differentiation are defined for formal power series (see Niven's paper for details).

(a) Recall the *divisor function*, which is defined by $\sigma(n) := \sum_{d|n} d$. For example, $d(6) = 1+2+3+6 = 12$. Prove that

$$\sum_{n\geq 1}\frac{nq^n}{1-q^n} = \sum_{n\geq 1}\sigma(n)q^n.$$

(b) Let $P(q) := \frac{1}{(q;q)_\infty} = \sum_{n\geq 0} p(n)q^n$. Calculate the logarithmetic derivative:

$$\frac{1}{P(q)}\cdot\frac{d}{dq}P(q) = \frac{d}{dq}\log(P(q)).$$

How does this compare to part (a)?

(c) Prove a result of Euler, that for $n \geq 1$,

$$np(n) = \sum_{k=1}^{n}\sigma(k)p(n-k).$$

Remark: A similar argument, using $P(q)^{-1}$ in place of P and applying the the Pentagonal Number Theorem, gives another identity of Euler:

$$\sum_{k\in\mathbb{Z}}(-1)^k\sigma\left(n-\frac{k(3k-1)}{2}\right)=\begin{cases}(-1)^{m-1}\cdot\dfrac{m(3m-1)}{2} & \text{if } n=\dfrac{m(3m-1)}{2};\\ 0 & \text{otherwise.}\end{cases}$$

Problems 6–8 address Jacobi's Triple Product identity. We will primarily use the second form, which states

$$\sum_{n\in\mathbb{Z}}(-1)^n z^n q^{\frac{n(n-1)}{2}}=\left(z,z^{-1}q,q;q\right)_\infty.$$

6. (a) Show that if $z=1$ is substituted directly into the identity, then both sides are 0.

   (b) Verify that the left-hand side can be rewritten as

$$\sum_{n\geq 0}(-1)^n q^{\frac{n(n+1)}{2}}\left(z^{-n}-z^{n+1}\right).$$

   (c) Now divide by $(1-z)$ and take the limit as $z\to 1$. You should conclude the identity

$$\sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{n(n-1)}{2}}=\prod_{n\geq 1}(1-q^n)^3.$$

7. In this problem you will explore an alternative proof of Jacobi's Triple Product.

   (a) Use Euler's Theorem to obtain the double sum:

$$\left(z,z^{-1}q;q\right)_\infty=\sum_{n,m\geq 0}\frac{(-1)^{n+m}z^{n-m}q^{\frac{n(n-1)}{2}+\frac{m(m+1)}{2}}}{(q;q)_n(q;q)_m}.$$

   (b) Make the substitution $k=n-m$ and show that the above is equal to the following expression:

$$\sum_{k\geq 0}(-1)^k z^k q^{\frac{k(k-1)}{2}}\sum_{m\geq 0}\frac{q^{m^2+mk}}{(q;q)_{m+k}(q;q)_m}+\sum_{k<0}(-1)^k z^k q^{\frac{k(k-1)}{2}}\sum_{n\geq 0}\frac{q^{n^2+n(-k)}}{(q;q)_n(q;q)_{n+(-k)}}.$$

   (c) The proof concludes by applying the following identity, which holds for any $k\geq 0$:

$$\sum_{m\geq 0}\frac{q^{m(m+k)}}{(q;q)_{m+k}(q;q)_m}=\frac{1}{(q;q)_\infty}.$$

   You are not required to prove this, but you are encouraged to try! One method is to make an appropriate substitution in Cor. 2.6 from Andrews. Another is to use a *Durfee rectangle* decomposition of partitions; this is a modification of the proof at the end of Chapter 2.

8. This problem gives another proof of Jacobi's Triple Product. Let $F(z;q):=\left(z,z^{-1}q,q;q\right)_\infty$ denote the product side.

(a) Prove that $F(z;q) = -zF(zq;q)$ and $F(z;q) = F(z^{-1}q;q)$.

(b) Consider the Laurent expansion in z, so

$$F(z;q) = \sum_{n\in\mathbb{Z}} a_n(q) z^n.$$

Use the properties from part (a) to conclude that $a_n = -q^{n-1}a_{n-1}$ for $n > 0$, and $a_{-n} = q^n \cdot a_n$.

(c) Finally, show that

$$F(z;q) = a_0(q) \sum_{n\in\mathbb{Z}} (-1)^n z^n q^{\frac{n(n-1)}{2}}.$$

Remark: It is actually still some work to show that $a_0(q) = 1$; note that it is not possible to plug in $z = 0$!

9. A famous class of partitions was studied by Rogers and Ramanujan. Let $\mathcal{RR}$ denote the set of partitions λ into distinct parts such that no consecutive integers appear as parts; i.e.,

$$\mathcal{RR} := \left\{\lambda \in \mathcal{P} \mid \lambda_i \geq \lambda_{i+1} + 2 \text{ for } 1 \leq i \leq \ell(\lambda)\right\}.$$

(a) Prove the generating function identity

$$R(x;q) := \sum_{\lambda\in\mathcal{RR}} x^{\ell(\lambda)} q^{|\lambda|} = \sum_{n\geq 0} \frac{x^n q^{n^2}}{(q;q)_n}.$$

Hint: Recall the combinatorial proof of Euler's second identity (Andrews (2.2.6)), where a diagonal "staircase" of parts of size $1, 2, \ldots, \ell$ was removed. For Rogers-Ramanujan partitions, consider a staircase of parts $1, 3, 5, \ldots, 2\ell - 1$.

(b) Prove the q-difference equation

$$R(x;q) - R(xq;q) = xqR(xq^2;q).$$

There are multiple ways to approach this; you can work directly with the series from part (a), or you can also think combinatorially and consider whether or not 1 occurs in $\lambda \in \mathcal{RR}$.

10. Recall the notation for basic hypergeometric series:

$${}_2\phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; q, t\right) := \sum_{n\geq 0} \frac{(a,b;q)_n t^n}{(q,c;q)_n}.$$

Heine's transformation (Cor. 2.3 in Andrews) is then written as

$${}_2\phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; q, t\right) = \frac{(at,b;q)_\infty}{(c,t;q)_\infty} {}_2\phi_1\left(\begin{matrix} \frac{c}{b}, & t \\ & at \end{matrix}; q, b\right).$$

In fact, Heine and Rogers also found two other related transformations:

$${}_2\phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; q, t\right) = \frac{\left(\frac{c}{b}, bz; q\right)_\infty}{(z,c;q)_\infty} {}_2\phi_1\left(\begin{matrix} b, & \frac{abz}{c} \\ & bz \end{matrix}; q, \frac{c}{b}\right); \tag{1}$$

$${}_2\phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; q, t\right) = \frac{\left(\frac{abz}{c}; q\right)_\infty}{(z;q)_\infty} {}_2\phi_1\left(\begin{matrix} \frac{c}{a}, & \frac{c}{b} \\ & c \end{matrix}; q, \frac{abz}{c}\right). \tag{2}$$

Prove at least one of these.

Remark: Note that the original function and the right side of (2) *are clearly symmetric in a and b, but the other expressions are not…this leads to a very short proof that links all three, although you can also just prove each identity directly.*