MATH 7230 Homework 4 - Spring 2017

Due Thursday, Mar. 2 at 10:30

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You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

1. As proved in lecture, the Chu-Vandermonde identity states that

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}. \tag{1}$$

This is sometimes written alternatively as a hypergeometric summation formula. Recall that Gauss' hypergeometric series is defined by

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} ; t \right) := \sum_{n \geq 0} \frac{(a)_n (b)_n t^n}{(c)_n n!},$$

where the Pochhammer symbol is $(x)_n := x \cdot (x+1) \cdots (x+n-1)$. The Chu-Vandermonde identity is then

$${}_2F_1\left(\begin{matrix} -N, & -b \\ & c \end{matrix} ; 1 \right) = \frac{(c+b)_N}{(c)_N}. \tag{2}$$

Here k is a nonnegative integer, and b and c are arbitrary. Prove that the two formulations (1) and (2) are equivalent.

Problems 2–4 explore some properties of q-calculus and the relation to q-basic hypergeometric series.

2. The *q-derivative* of a function $f(x)$ is defined by

$$D_q(f(x)) := \frac{f(qx) - f(x)}{(q-1)x}.$$

Note that as $q \to 1$, this coincides with the usual definition of the derivative.

(a) Prove that $D_q(x^n) = [n]_q \cdot x^{n-1}$, where $[n]_q := \dfrac{1-q^n}{1-q}$ is a *q-integer*.

(b) Prove that product rule for the q-derivative:

$$D_q(fg(x)) = f(qx) \cdot D_q(g(x)) + D_q(f(x)) \cdot g(x).$$

(c) Define the q-analog of a binomial power as $(x-a)_q^n := \prod_{j=1}^{n}\left(x-aq^j\right)$. Prove that

$$D_q\left((x-a)_q^n\right) = [n]_q \cdot (x-a)_q^{n-1}.$$

Remark: This result leads to q-Taylor's theorem:

$$f(x) = \sum_{n\geq 0} D_q^n(f(x))\Big|_{x=a} \cdot \frac{(x-a)_q^n}{[n]_q!}.$$

3. Jackson's indefinite q-*integral* is defined by

$$I_q(f(x)) = \int f(x)d_q(x) := (1-q)x \cdot \sum_{n\geq 0} q^n f\left(xq^n\right).$$

(a) Calculate $I_q(1)$.

(b) Prove that this is an anti-derivative, so that $D_q(F_q(f(x)) = f(x)$.

(c) The definite q-integral is defined similarly, by plugging in endpoints as usual:

$$\begin{aligned}\int_a^b f(x)d_q(x) &= (1-q)x \cdot \sum_{n\geq 0} q^k f\left(xq^n\right)\Bigg|_a^b \\ &= (1-q)a \cdot \sum_{n\geq 0} q^k f\left(aq^n\right) - (1-q)b \cdot \sum_{n\geq 0} q^k f\left(bq^n\right).\end{aligned}$$

Show that $\int_0^1 x^n d_q(x) = \frac{1}{[n]_q}$.

4. The classical *Gamma function* is defined (for $\operatorname{Re}(s) > 0$) by

$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt.$$

It satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s), \tag{3}$$

which is also used to extend Γ meromorphically. This function "interpolates" the factorial in the sense that $\Gamma(n+1) = n!$ for $n \geq 0$.

This is also an alternative expression as an infinite product, which is valid for all complex s excluding the negative integers:

$$\Gamma(s) = \frac{1}{s}\prod_{n\geq 1}\left(1+\frac{s}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^s.$$

It is this second definition that most naturally leads to the q-Gamma function, which is defined by

$$\Gamma_q(s) := (1-q)^{1-s}\frac{(q;q)_\infty}{(q^s;q)_\infty}.$$

(a) Verify that $\Gamma(s)$ satisfies (3), using either (or both) expressions.

(b) Prove the functional equation

$$\Gamma_q(s+1) = \frac{1-q^s}{1-q}\Gamma_q(s).$$

(c) Finally, in order to see that $\lim_{q\to 1}\Gamma_q(s) = \Gamma(s)$, prove that

$$\Gamma_q(s+1) = \prod_{n\geq 1} \frac{\left(1-q^{n+1}\right)^s}{\left(1-q^{s+n}\right)\left(1-q^n\right)^{s-1}}.$$

Evaluate the limit as $q \to 1$ termwise and recover the product formula for $\Gamma(s+1)$.

Remark: This is not a fully rigorous proof of $\lim_{q\to 1}\Gamma_q(s) = \Gamma(s)$, *as there are issues of convergence that were ignored.*

Recall that the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}.$$

5. Prove that $\displaystyle\lim_{n\to\infty} \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{1}{(q;q)_m}$.

6. The first identity of Andrews' Theorem 3.3 states that

$$(z;q)_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\frac{j(j-1)}{2}} z^j.$$

Prove this in at least **two** different ways:

- Induction;
- Special case of Cauchy's Theorem (Andrews Thm 2.1);
- Combinatorial, setting $z \mapsto -zq$, and considering the generating function for partitions into distinct parts of size at most n.

7. Andrews 3.1.