MATH 7230 Homework 5 - Spring 2017

Due Thursday, Mar. 9 at 10:30

`www.math.lsu.edu/~mahlburg/`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

1. As proven in lecture, the generating function for inversions of permutations has the closed-form
$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = \frac{(q;q)_n}{(1-q)^n}. \tag{1}$$
Here $\mathrm{inv}(\sigma)$ counts the number of pairs (i,j) such that $i < j$ and $\sigma_i > \sigma_j$. Apply the operator $\left.\frac{d}{dq}\right|_{q=1}$ to both sides of (1) in order to prove that the average value of $\mathrm{inv}(\sigma)$ is $\frac{n(n-1)}{4}$.

2. Let $\mathcal{P}_{N \times M}$ denote the set of partitions that fit in a box of size $N \times M$; i.e.,
$$\mathcal{P}_{N \times M} := \{\lambda \mid \alpha(\lambda) \leq N, \ell(\lambda) \leq M\},$$
where α is the largest part, and ℓ is the number of parts. Theorem 3.1 in Andrews states that
$$\sum_{\lambda \in \mathcal{P}_{N \times M}} q^{|\lambda|} = \begin{bmatrix} N+M \\ M \end{bmatrix}_q = \frac{(q;q)_{N+M}}{(q;q)_N (q;q)_M}. \tag{2}$$
Apply the operator $\left.\frac{d}{dq}\right|_{q=1}$ to (2) and determine the average size of partitions in $\mathcal{P}_{N \times M}$.

3. Use simple combinatorial arguments (exploiting symmetry properties!) to obtain the correct values for Problems 1 and 2.

4. Andrews 3.3. Use Euler's identity ((2.2.5) in Andrews) to expand both products as series in x, and then group terms.

5. The q-Chu-Vandermonde summation formula states
$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k-j \end{bmatrix}_q q^{(n-j)(k-j)}.$$
In this problem you will give a combinatorial proof of the identity.

(a) The left-hand side is the generating function partitions that lie in a box of row length $(n-k)+m$ and column length k. Explain why each such partition has a unique *Durfee rectangle* of dimensions $((n-k)+i) \times i$, where i is maximal.

Remark: This is still valid even if $(n-k)$ is negative! The sign simply affects whether the Durfee rectangle has longer columns than rows (positive case) or longer rows than columns (negative).

(b) Draw the appropriate pictures and explain the following argument: To the right of the Durfee rectangle is an arbitrary partition $\mu \in \mathcal{P}_{(m-i)\times i,}$, and below the rectangle is an arbitrary $\gamma \in \mathcal{P}_{((n-k)+i)\times(k-i)}$.

(c) Combine the components from part (b) to obtain the overall generating function. Replace $i \mapsto k-j$ for the final problem statement.