MATH 7230 Homework 6 - Spring 2017

Due Thursday, Mar. 23 at 10:30

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You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

1. As discussed in lecture, the generating function for the inversion statistic on multiset permutations is related to q-multinomial coefficients. In particular, if σ is a permutation (i.e., a reordering) of the multiset $\{1^{m_1}2^{m_2}\cdots r^{m_r}\}$, then the *inversion* statistic is defined by
$$\mathrm{inv}(\sigma) := \#\big\{(i,j) \ \mid \ i<j \text{ and } \sigma_i > \sigma_j\big\}.$$
Theorem 3.11 in Andrews states that for any permutation $\tau \in S_r$,
$$\sum_{\sigma \in \mathrm{Sym}(1^{m_1}2^{m_2}\cdots r^{m_r})} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathrm{Sym}(1^{m_{\tau(1)}}2^{m_{\tau(2)}}\cdots r^{m_{\tau(r)}})} q^{\mathrm{inv}(\sigma)}.$$
In this problem you will (partially) explore the combinatorial proof mentioned in lecture.
   (a) First, prove the theorem for the case $r=2$. If $\sigma \in \mathrm{Sym}(1^m 2^n)$, first replace all 1's by 2's, and all 2's by 1's. Then define σ' by reversing the permutation. Precisely,
   $$\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_{m+n},$$
   where $\sigma'_i := 3 - \sigma_{m+n+1-i}$. Prove that $\sigma \mapsto \sigma'$ is a bijection from $\mathrm{Sym}(1^m2^n)$ to $\mathrm{Sym}(1^n2^n)$, and $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma')$.
   (b) Theorem 3.5 in Andrews uses partition diagrams to prove that
   $$\sum_{\sigma \in \mathrm{Sym}(1^m2^n)} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} n+m \\ n,\, m\end{bmatrix}_q.$$
   In particular, a multiset permutation σ is associated to an edge path that traces out the lower boundary of a partition $\lambda \in \mathcal{P}_{n\times m}$. Provide an alternative proof of part (a) by considering the conjugate partition λ'.
   (c) Prove the case
   $$\sum_{\sigma \in \mathrm{Sym}(1^{m_1}2^{m_2}3^{m_3})} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathrm{Sym}(1^{m_2}2^{m_1}3^{m_3})} q^{\mathrm{inv}(\sigma)}.$$
   In particular, given $\sigma \in \mathrm{Sym}(1^{m_1}2^{m_2}3^{m_3})$, fix the position of the 3's, and apply the procedure from part (a) to the 1's and 2's to obtain $\sigma' \in \mathrm{Sym}(1^{m_2}2^{m_1}3^{m_3})$. Complete the proof by arguing that $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma')$.
   Remark: As mentioned in lecture, this procedure can be iterated to prove the general case, using the fact that any permutation $\tau \in S_r$ can be written as a product of adjacent transpositions $(i,\, i+1)$.

Problems 2 – 4 lead to an alternative proof of Theorem 3.1 from Andrews (which states that the q-binomial coefficient $\left[{N+M \atop M} \right]_q$ generates restricted partitions in $\mathcal{P}_{N\times M}$). Be sure to use **only** the combinatorial/inductive arguments outlined below, and do **not** appeal to Theorem 3.1.

2. In this problem you will answer Andrews 3.13 by carefully thinking about generating functions and parts. Denote the partitions under consideration by

$$\mathcal{E}_{2,j} := \big\{\lambda \mid \ell(\lambda) = j,\ \lambda_1 \le \lambda_2 + i\big\}.$$

Let $\mathcal{P}_j$ be the set of partitions with exactly j parts, with generating function $P_j(q) := \sum_{\lambda\in\mathcal{P}_j} q^{|\lambda|} = \dfrac{q^j}{(q;q)_j}$.

(a) Show that the complement of $\mathcal{E}_{2,j}$ in $\mathcal{P}_j$ is

$$\mathcal{E}^c_{2,j} = \big\{\lambda \mid \ell(\lambda) = j,\ \lambda_1 \ge \lambda_2 + i + 1\big\}.$$

(b) Suppose that $\lambda \in \mathcal{E}^c_{2,j}$, with parts $\lambda_1, \lambda_2, \ldots, \lambda_j$. Define $\widetilde{\lambda}$ by setting $\widetilde{\lambda}_1 := \lambda_1 - i - 1$, and $\widetilde{\lambda}_r := \lambda_r$ for $2 \le r \le j$. Show that the map $\lambda \mapsto \widetilde{\lambda}$ is a bijection from $\mathcal{E}^c_{2,j}$ to $\mathcal{P}_j$.

(c) Translate part (b) into the generating function identity

$$\sum_{\lambda\in\mathcal{E}^c_{2,j}} q^{|\lambda|} = q^{i+1} P_j(q).$$

Conclude the claimed result from the problem statement, that

$$\sum_{\lambda\in\mathcal{E}_{2,j}} q^{|\lambda|} = \left(1 - q^{i+1}\right) P_j(q).$$

3. In this problem you will answer Andrews 3.14. The general approach will be an inductive argument on k. Note that Problem 2 is the case $k = 1$.

For $1 \le k \le j-1$ (indexed to be consistent with Andrews), define

$$\mathcal{E}_{k+1,j} := \big\{\lambda \mid \ell(\lambda) = j,\ \lambda_1 \le \lambda_{k+1} + i\big\}.$$

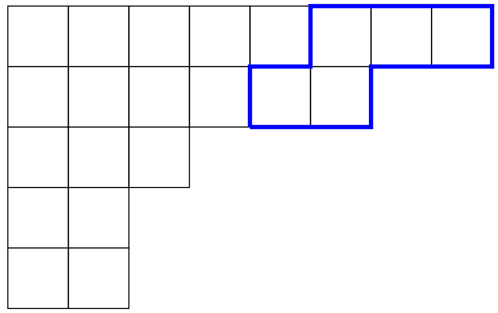
Denote the corresponding generating functions by

$$E_{k+1,j}(q) := \sum_{\lambda\in\mathcal{E}_{k+1,j}} q^{|\lambda|}.$$

(a) First consider the case $k = 2$. Show that $\mathcal{E}_{3,j} \subseteq \mathcal{E}_{2,j}$, with complement (relative to $E_{2,j}$)

$$\mathcal{E}^c_{3,j} = \big\{\lambda \mid \ell(\lambda) = j,\ \lambda_1 - \lambda_2 \le i,\ (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) \ge i + 1\big\}.$$

Figure 1: A hook of length 5 in the partition $\lambda = 8 + 6 + 3 + 2 + 2$.



(b) Suppose that $\lambda \in \mathcal{E}^c_{3,j}$, and define $\widetilde{\lambda}$ by removing a λ_1–λ_2 *hook* of length $i+2$. Algebraically, this is achieved by setting $\widetilde{\lambda}_1 := \lambda_2 - 1$, $\widetilde{\lambda}_2 := \lambda_1 - i - 1$, and $\widetilde{\lambda}_r := \lambda_r$ for $3 \le r \le j$. Note that $|\widetilde{\lambda}| = |\lambda| - i - 2$.

This definition is most easily understood by visualizing the "hook" along the bottom layer of blocks in the partition diagram. See Figure 1, which illustrates the case that $\lambda = 8 + 6 + 3 + 2 + 2$. The λ_1–λ_2 hook of length 5 is outlined in blue, and once it is removed, the remaining partition is $\widetilde{\lambda} = 5 + 4 + 3 + 2 + 2$.

Show that the map $\lambda \mapsto \widetilde{\lambda}$ is a bijection from $\mathcal{E}^c_{3,j}$ to $\mathcal{E}_{2,j}$. Conclude the generating series identity

$$E_{3,j}(q) = \left(1 - q^{i+2}\right) E_{2,j}(q).$$

By Problem 2, $E_{3,j}(q) = \left(1 - q^{i+2}\right)\left(1 - q^{i+1}\right) P_j(q)$, as desired.

(c) Similarly, suppose that the formula has been proved for $E_{k,j}(q)$, and consider $\lambda \in \mathcal{E}^c_{k+1,j}$. Define $\widetilde{\lambda}$ by removing a λ_1–λ_2–$\cdots$–λ_{k+1} hook of length $i+k$. Draw the appropriate picture and/or write out the definition of $\widetilde{\lambda}$ precisely.

Conclude that

$$E_{k+1,j}(q) = \left(1 - q^{i+k}\right) E_{k,j}(q).$$

4. Andrews 3.15. In particular, show that Theorem 3.1 follows from the case $k = j - 1$, which shows that

$$E_{k,j}(q) = \frac{\left(q^{i+1}; q\right)_j q^j}{(q;q)_j}.$$

Now suppose that $\lambda \in \mathcal{E}_{k,j}$, and define μ by $\mu_r := \lambda_r - \lambda_j$ for $1 \le r \le j-1$. Show that $\mu \in \mathcal{P}_{i\times(j-1)}$. Conclude that $E_{k,j}(q) = \dfrac{q^j}{1-q^j} \cdot G_{i\times(j-1)}(q)$.

An important tool in asymptotic analysis is the ability to estimate integrals using the Method of Steepest Descent. In Problems 5–6 you will use this technique to prove Stirling's formula for the factorial (or Gamma) function,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \qquad (1)$$

See K. Conrad's notes for more details:
(`http://www.math.uconn.edu/~kconrad/blurbs/analysis/stirling.pdf`).

5. (a) Recall that the Gamma function is defined by $\Gamma(s+1) = \int_0^\infty e^{-t} t^s dt$. Suppose that $s \geq -1$, and consider the integrand as a function of t, $f(t) = e^{-t}t^s$. Prove that $f(t)$ has a maximum at $t = s$. Also prove that $f(t)$ has inflection points at $t = s \pm \sqrt{s}$; the fact that these are symmetric around the peak is not essential to the general technique, but it does strongly suggest that the bulk of the integral's contribution occurs in the interval $[s - \sqrt{s}, s + \sqrt{s}]$.

   (b) Following the observations above, make the change of variables $t \mapsto s + \sqrt{s}t$, and fill in the details to obtain the formula

$$\Gamma(s+1) = \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\sqrt{s}}^\infty \left(1 + \frac{t}{\sqrt{s}}\right)^s e^{-\sqrt{s}t} dt.$$

   (c) Calculate the Taylor expansion and write the integrand as

$$\exp\left(s \log\left(1 + \frac{t}{\sqrt{s}}\right) - \sqrt{s}t\right) = \exp\left(\sum_{k \geq 2} (-1)^{k-1} \frac{t^k}{k s^{\frac{k}{2}-1}}\right).$$

   Derive Stirling's formula by approximating the integral using only the $k = 2$ term, and then evaluating the Gaussian integral:

$$\begin{aligned}\Gamma(s+1) &= \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\sqrt{s}}^\infty \exp\left(\sum_{k \geq 2} (-1)^{k-1} \frac{t^k}{k s^{\frac{k}{2}-1}}\right) dt \\ &\sim \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\sqrt{s}}^\infty e^{-\frac{t^2}{2}} dt \sim \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\infty}^\infty e^{-\frac{t^2}{2}} dt. \qquad (2)\end{aligned}$$

   You will fill in the details of the approximations in Problem 6.

6. (a) The final approximation in (2) requires that the integral from $-\infty$ to $-\sqrt{s}$ be negligible asymptotically. In fact, you will now show that it is *exponentially* small. In particular, consider

$$\int_{-\infty}^{-\sqrt{s}} e^{-\frac{t^2}{2}} dt,$$

   and make the substitution $t = -\sqrt{u}$. You should obtain

$$\int_s^\infty e^{-\frac{u}{2}} \frac{du}{2\sqrt{u}}.$$

   Now bound the integrand using $e^{-\frac{u}{2}} \frac{1}{\sqrt{u}} < e^{-\frac{u}{2}}$, and evaluate.

   (b) The first approximation in (2) follows from Taylor's Theorem with Remainder, which shows that $\log(1+x) = x - \frac{x^2}{2} + O(x^3)$. In fact, the constant here is uniform, so that

$$\left|\log(1+x) - x + \frac{x^2}{2}\right| < cx^3$$

for some c. Then

$$\Gamma(s+1) = \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\sqrt{s}}^{\infty} e^{-\frac{t^2}{2}} \cdot e^{O\left(\frac{t^3}{\sqrt{s}}\right)} dt$$
$$= \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\sqrt{s}}^{\infty} e^{-\frac{t^2}{2}} \cdot \left(1 + O\left(\frac{t^3}{\sqrt{s}}\right)\right) dt. \quad (3)$$

Use integration by parts to show that

$$\int_0^{\infty} t^3 e^{-\frac{t^2}{2}} dt = 2 \int_0^{\infty} t e^{-\frac{t^2}{2}} dt = 2,$$

and conclude that the big-O term in (3) contributes

$$\left(\frac{s}{e}\right)^s \sqrt{s} \cdot \frac{1}{\sqrt{s}} \int_{-\sqrt{s}}^{\infty} O\left(e^{-\frac{t^2}{2}} t^3\right) dt = O\left(\left(\frac{s}{e}\right)^s\right).$$

In particular, this is $\frac{1}{\sqrt{s}}$ smaller than the main term in Stirling's formula, which is a *polynomial* error.

Remark: Working with more terms in the Taylor expansion gives an "asymptotic expansion" with higher precision; for example,

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right).$$

7. Recall that the Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

   (a) Let $c_{12}(n)$ denote the number of compositions of size n where all of the parts are 1 or 2. Prove that $c_{12}(n) = F_{n+1}$.

   Hint: Show that $c_{12}(n) = c_{12}(n-1) + c_{12}(n-2)$ by separating the cases that the first part is 1 or 2.

   (b) Andrews 4.1. Let $c_{>1}(n)$ denote the number of compositions of size n where all of the parts are larger than 1. Suppose that $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ is such a composition. Show that if $\mu_1 > 2$, then $(\mu_1 - 1, \mu_2, \ldots, \mu_\ell)$ is counted by $c_{>1}(n-1)$; if $\mu_1 = 2$, then $(\mu_2, \ldots, \mu_\ell)$ is counted by $c_{>1}(n-2)$. Conclude that $c_{>1}(n) = c_{>1}(n-1) + c_{>1}(n-2)$, and finally show that $c_{>1}(n) = F_{n-1}$.

8. In Problem 7 you proved that for $n \geq 1$, $c_{12}(n)$ and $c_{>1}(n+2)$ are both equal to F_{n+1}. Give a direct bijective proof that $c_{12}(n) = c_{>1}(n+2)$.

9. In this problem you will prove the Ramanujan congruence $p(7n+5) \equiv 0 \pmod 7$. The argument is similar to the modulo 5 congruence, which is outlined in Andrews 10.7 – 10.13. Begin by writing

$$\sum_{n \geq 0} p(n) q^n = \frac{1}{(q;q)_\infty} = (q;q)_\infty^3 \cdot (q;q)_\infty^3 \cdot \frac{1}{(q;q)_\infty^7}.$$

   (a) Show that $\frac{1}{(q;q)_\infty^7} \equiv \sum_{r \geq 0} p(r) q^{7r} \pmod 7$.

(b) Recall the identity from Homework 3, Problem 6:

$$(q;q)_\infty^3 = \sum_{m\geq 0}(-1)^m(2m+1)q^{\frac{m(m+1)}{2}}.$$

By considering all possible values of m mod 7, show that

$$(q;q)_\infty^3 \cdot (q;q)_\infty^3 \equiv \sum_{k\not\equiv 5 \pmod 7} a(k)q^k \pmod 7,$$

where $a(k)$ are certain coefficients (the key point is that there are no terms of the form q^{7k+5}).

(c) Conclude that

$$\sum_{n\geq 0} p(n)q^n \equiv \sum_{n\not\equiv 5 \pmod 7} c(n)q^n \pmod 7.$$