MATH 7230 Homework 7 - Spring 2017

Due Thursday, Apr. 6 at 10:30

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You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

In Problems 1 – 2 you will fill in some of the details in the theory of formal infinite product representations of formal power series.

1. (a) Suppose that $f(q) = 1 + \sum_{n\geq 1} a(n)q^n \in \mathbb{Z}[[q]]$ is a formal power series with integer coefficients. We proved in class that there is a corresponding formal power series

$$f(q) = \prod_{m\geq 1} (1-q^m)^{b(m)},$$

where $b(m) \in \mathbb{Z}$. Rewrite the proof in your own words (and fill in any missing details!).

(b) Now consider the case that f is not monic, so $f(q) = \sum_{n\geq 0} a(n)q^n \in \mathbb{Z}[[q]]$. Prove that the formal power series has the form

$$f(q) = a(0) \prod_{m\geq 1} (1-q^m)^{b(m)},$$

where now $b(m) \in \mathbb{Q}$.

(c) A much more general case is

$$f(q) = \sum_{n\geq N} a(n)q^{n+r},$$

where $N \in \mathbb{Z}, r \in \mathbb{R}$, and $a(n) \in \mathbb{R}$. Use the generalized Binomial Theorem to prove that the formal product has the form

$$f(q) = c\, q^s \prod_{m\geq 1} (1-q^m)^{b(m)},$$

where $s, c, b(m) \in \mathbb{R}$.

Remark: The case that $a(n), r \in \mathbb{Q}$ is most relevant for algebraic applications, such as Borcherds' proof of the Monstrous Moonshine Conjecture.

2. Calculate the exponents (out to the given power of q) of the formal infinite product expansions for each of the following series. You may do this by hand, or by finding/programming a computational routine, or even by recognizing a known identity from the literature!

(a) $1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+5q^9+\cdots$;
(b) $1+q^2+q^3+q^4+q^5+2q^6+2q^7+3q^8+3q^9+\cdots$;
(c) $1+2q^2+O\left(q^{11}\right)$;
(d) $1+2q+3q^2+3q^3+3q^4+3q^5+3q^6+q^7-q^8-3q^9-3q^{10}+\cdots$.

In Problems 3 – 4 you will prove Lehmer's result on partitions with gap conditions, which states that there are no "simple" infinite product identities for partitions whose parts differ by at least 3.

3. A partition has *d-gaps* if $\lambda_i-\lambda_{i+1}\geq d$ for all i. For example, the case $d=1$ is partitions into distinct parts, and $d=2$ is the Rogers-Ramanujan partitions. Let $q_d(n)$ count the number of partitions of n with d-gaps. Prove that

$$f_d(q):=\sum_{n\geq 0} q_d(n)q^n=\sum_{n\geq 0}\frac{q^{\frac{dn(n-1)}{2}+n}}{(q;q)_n}. \tag{1}$$

4. Lehmer proved that $f_d(q)$ cannot equal a product of the form $\frac{1}{(1-q^{a_1})(1-q^{a_2})\cdots}$. His proof proceeds by contradiction, considering the first several terms of (1). Suppose to the contrary that there are indeed $a_1<a_2<\ldots$ such that

$$\prod_{j\geq 1}\frac{1}{1-q^{a_j}}=1+\frac{q}{1-q}+\frac{q^{d+2}}{(1-q)(1-q^2)}+\frac{q^{3d+3}}{(1-q)(1-q^2)(1-q^3)}+O\left(q^{6d+4}\right). \tag{2}$$

(a) Show that $a_1=1$, and $a_2=d+2$ (recalling Problem 1 if you solved it, although it is not necessary!).

(b) Multiply (2) by $(1-q)(1-q^{d+2})$, obtaining

$$\prod_{j\geq 3}\frac{1}{1-q^{a_j}}=1-q^{d+2}+\frac{q^{d+2}\left(1-q^{d+2}\right)}{1-q^2}+\frac{q^{3d+3}\left(1-q^{d+2}\right)}{\left(1-q^2\right)\left(1-q^3\right)}+O\left(q^{6d+4}\right). \tag{3}$$

Observe that all coefficients on the left side of (3) are non-negative.

(c) If d is odd, obtain a contradiction by showing that the coefficient of q^{2d+4} on the right side of (3) is negative.

(d) Otherwise, if d is even, show that $a_j=d+2j-2$ for $2\leq j\leq\frac{d}{2}+2$ (note that you have already found the $j=2$ value). In particular, multiply (3) by $\left(1-q^{d+4}\right)$, and then consider the resulting coefficient of q^{2d+8} to obtain a contradiction. You will need to take some care to ensure that the q^{3d+3} term does not interfere....

Remark: Let $\mathcal{Q}_d$ denote the partitions with d-gaps. Alder extended Lehmer's result by showing that there is also no simple infinite product representation for $\mathcal{Q}_{d,\geq k}$ (parts at least k).

Problems 5 – 6 address some of the power series manipulations required for the proof of Schur's theorem from lecture. Supplementary definitions and notation:

- If $\lambda=(\lambda_1,\ldots,\lambda_\ell)\in\mathcal{P}$, define the "$+k$" map by adding k to each part, and denote it by

$$\lambda+k:=(\lambda_1+k,\ldots,\lambda_\ell+k).$$

- For a subset of partitions $\mathcal{A} \subset \mathcal{P}$, denote the generating function by

$$f_{\mathcal{A}}(x;q) := \sum_{\lambda \in \mathcal{A}} x^{\ell(\lambda)} q^{|\lambda|}.$$

Furthermore, let $\mathcal{A}_{\geq k} := \{\lambda \in \mathcal{A} \mid \lambda_i \geq k \; \forall \, i\}$, and define $\mathcal{A}_{>k}$ analogously.

- For $k \in \mathbb{N}$, $\mathcal{A} \subset \mathcal{P}$ is *strongly closed* under "$+k$" if $\lambda \mapsto \lambda + k$ gives a bijection $\mathcal{A} \leftrightarrow \mathcal{A}_{>k}$ (with inverse map $\lambda \mapsto \lambda - k$). We say that $\mathcal{A}$ is strongly closed under $\mathcal{K} \subset \mathbb{N}$ if it is strongly closed for all $k \in \mathcal{K}$.

5. (a) Suppose that $\mathcal{A}$ is *strongly closed* under $\mathcal{K}$. If $k \in \mathcal{K}$, prove that

$$f_{\mathcal{A}_{>k}}(x;q) = f_{\mathcal{A}_{\geq k+1}}(x;q) = f_{\mathcal{A}}(xq^k;q).$$

(b) Verify that the set of all partitions $\mathcal{P}$ is strongly closed under $\mathbb{N}$. Conclude that

$$f_{\mathcal{P}_{\geq k}}(x;q) = \frac{1}{(xq^k;q)_\infty}.$$

6. (a) Recall from Homework 3 Problem 9 that the Rogers-Ramanujan partitions are defined by

$$\mathcal{RR} := \{\lambda \in \mathcal{P} \mid \lambda_i \geq \lambda_{i+1} + 2 \text{ for } 1 \leq i \leq \ell(\lambda)\}.$$

Prove that $\mathcal{RR}$ is strongly closed under $\mathbb{N}$.

Remark: In fact, the generating functions for $\mathcal{RR}$ and $RR_{\geq 2}$ (with $x = 1$) are seen above in Problem 2 (a) and (b).

(b) The *Schur (gap 3)* partitions are defined by

$$\mathcal{S} := \left\{ \lambda \;\middle|\; \lambda_i - \lambda_{i+1} \geq \begin{cases} 4 & \text{if } 3 \mid \lambda_i \\ 3 & \text{otherwise,} \end{cases} \quad \forall \, i \right\}.$$

Show that $\mathcal{S}$ is strongly closed under $3\mathbb{N}$.

Remark: This was used in the proof of Schur's theorem, as combined with Problem 5 it implies that $f_{\mathcal{S}_{j+3}}(x;q) = f_{\mathcal{S}_j}(xq^3;q)$.

7. A key final step in Andrews' proof of Schur's partition theorem required Appell's Comparison Theorem. This states that if a (complex) sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit a, then

$$\lim_{x \to 1^-} (1-x) \cdot \sum_{n \geq 0} a_n x^n = a.$$

(a) Prove Appell's Comparison Theorem using the analytic definition of limit. The notation $x \to 1^-$ indicates that x approaches 1 from below on the real line (or in this case, along any path inside the complex unit circle) - why is this necessary?

Remark: Is there a version of Appell's Comparison Theorem for formal power series rather than the analytic statement?

(b) As a simple example, apply Appell's Comparison Theorem to evaluate $f(1)$ where

$$f(x) = f(x;q) := (1-x)\sum_{n\geq 0} \frac{x^n}{(q;q)_n}.$$

Compare to the expansion of $\dfrac{1}{(x;q)_\infty}$ provided by Euler's Identity (Andrews (2.2.5)).

Problems 8 – 10 prove some initial facts about the sum-product identity attributed to Göllnitz and Gordon's independent work in the 1960s (although the analytic form was shown earlier by Slater, and essentially the same identities are found in Ramanujan's notebooks).

8. Define the set of *Göllnitz-Gordon partitions* with the gap-2 condition given by

$$\mathcal{GG} := \left\{ \lambda \mid \lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } 2 \mid \lambda_i \\ 2 & \text{otherwise} \end{cases} \quad \forall i \right\}$$

(a) Explain why the condition for $\lambda \in \mathcal{GG}$ can be equivalently stated as distinct parts such that no two parts are consecutive integers, and all even parts differ by at least 4.

(b) Recalling Problem 5, show that $\mathcal{GG}$ is strongly closed under $2\mathbb{N}$.

9. In this problem you will derive the "sum-side" of the Göllnitz-Gordon identity. Denote the generating function and enumeration functions for Göllnitz-Gordon gap-2 partitions by

$$f_{\mathcal{GG}}(x) = f_{\mathcal{GG}}(x;q) = \sum_{m,n\geq 0} g(m,n)x^m q^n := \sum_{\lambda\in\mathcal{GG}} x^{\ell(\lambda)} q^{|\lambda|}.$$

Furthermore, let $g_j(m,n)$ only count those $\lambda \in \mathcal{GG}$ with m parts, size n, and smallest part at least j; denote the corresponding generating functions by $f_j(x)$ (note that $f_1(x) = f_{\mathcal{GG}}(x)$).

(a) Prove the system of recurrences:

$$\begin{aligned} g_1(m,n) &= g_2(m,n) + g_3(m-1,n-1), \\ g_2(m,n) &= g_3(m,n) + g_5(m-1,n-2). \end{aligned}$$

(b) Using the fact that $f_{j+2}(x) = f_j(xq^2)$ (cf. Problem 8 part (b) and Problem 8 part (a)), prove the q-difference equation

$$f_1(x) = (1+xq)f_1(xq^2) + xq^2 f_1(xq^4). \tag{4}$$

The direct proof proceeds by working with the corresponding generating functions from part (a) and solving the system to isolate only f_1 terms. If you prefer, you may provide a direct combinatorial proof of the q-difference equation.

(c) Expand (4) in powers of x, writing $f_1(x;q) = \sum_{m\geq 0} \alpha_m(q)x^m$. Solve for α_m and conclude the hypergeometric representation

$$f_{\mathcal{GG}}(x;q) = \sum_{n\geq 0} \frac{x^n q^{n^2}\left(-q;q^2\right)_n}{\left(q^2;q^2\right)_n}. \tag{5}$$

Remark: The "product-side" states that

$$f_{\mathcal{GG}}(1;q) = \left(q,q^4,q^7;q^8\right)_\infty^{-1},$$

i.e. the Göllnitz-Gordon gap-2 partitions are equinumerous with partitions into parts that are $1,4,7 \pmod 8$.

10. In this problem you will provide a combinatorial proof of (5). The 2-*modular diagram* of a partition is constructed by writing a row λ_i as $\left(\lceil \frac{\lambda_i}{2} \rceil - 1\right)$ 2's, followed by the remainder $\lambda_i \pmod 2 \in \{1,2\}$ (see Examples 1.6 – 1.7 in Andrews). For example, the partition $\lambda = 18+14+11+9+6+2 \in \mathcal{GG}$ (which will be a running example throughout this problem) has 2-modular diagram

2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2		
2	2	2	2	2	1			
2	2	2	2	1				
2	2	2						
2								

(a) Suppose that $\lambda \in \mathcal{GG}$ with $\ell(\lambda) = \ell$. Explain why λ must have an "odd ℓ-triangle" of size ℓ. This means that $\lambda_i \geq 2(\ell - i) + 1$ for $1 \leq i \leq \ell$; this can be viewed as a triangle of side length ℓ in the 2-modular diagram by splitting the 2's along the boundary of the triangle into two 1's:

2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2		
2	2	2	2	2	1			
2	2	2	2	1				
2	2	2						
2								

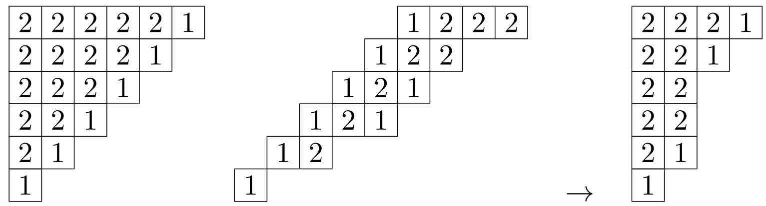
$\rightarrow$

2	2	2	2	2	11	2	2	2
2	2	2	2	11	2	2		
2	2	2	11	2	1			
2	2	11	2	1				
2	11	2						
11								

And then separating the 1's into distinct boxes, giving a triangle with odd rows:

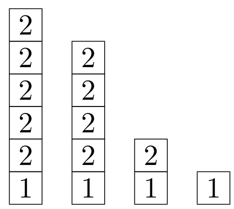
2	2	2	2	2	1	1	2	2	2
2	2	2	2	1	1	2	2		
2	2	2	1	1	2	1			
2	2	1	1	2	1				
2	1	1	2						
1	1								

(b) Now detach the remaining parts of each row, and slide the 1's to the right, combining with any previous 1's from the 2-modular diagram (the triangle is removed in the second part of the figure):



Explain why the gap-2 condition is now equivalent to the property that these excess rows form a partition $\widetilde{\lambda}$ with $\ell(\widetilde{\lambda}) = \ell$ that has arbitrary even parts and distinct odd parts.

(c) Read the columns of $\widetilde{\lambda}$ (i.e., take its conjugate $\widetilde{\lambda}'$) to obtain a partition with distinct odd parts, and all parts at most 2ℓ. For the running example the columns are



so $\widetilde{\lambda}' = 11 + 9 + 3 + 1$. Conclude (5) by combining the ℓ-triangle and $\widetilde{\lambda}$, and then summing over ℓ.