MATH 7230 Homework 8 - Spring 2017

Due Thursday, Apr. 20 at 10:30

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You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

In Problems 1 – 6 you will work through the details of Andrews and Baxter's paper: "A motivated proof of the Rogers-Ramanujan identities", American Mathematical Monthly **96** (1999), 401 – 409.

As on previous assignments, let $\mathcal{RR}$ denote the set of gap-2 partitions, where $\lambda_i - \lambda_{i+1} \geq 2$ for all i, with generating function

$$f_{\mathcal{RR}}(x;q) := \sum_{\lambda \in \mathcal{RR}} x^{\ell(\lambda)} q^{|\lambda|}.$$

On Homework 3 Problem 9 you showed that

$$f_{\mathcal{RR}}(x;q) = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q;q)_n}. \tag{1}$$

Furthermore, on Homework 7 Problem 6 you showed that $\mathcal{RR}$ is strongly closed under $\mathbb{N}$, so it makes sense to define

$$R_i(q) := f_{\mathcal{RR}}(q^{i-1};q) = \sum_{\lambda \in \mathcal{RR}_{\geq i}} q^{|\lambda|}.$$

The Rogers-Ramanujan identities then state that

$$R_1(q) = \sum_{\lambda \in \mathcal{RR}} q^{|\lambda|} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{2}$$

$$R_2(q) = \sum_{\lambda \in \mathcal{RR}_{\geq 2}} q^{|\lambda|} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{3}$$

1. Define the *Rogers-Ramanujan polynomials* for $i \geq 2$ by

$$A_i(q) := \sum_{\substack{\lambda \in \mathcal{RR} \\ \lambda_1 \leq i-2}} q^{|\lambda|}.$$

Note that these are in fact polynomials (why?). The reason for the shifted indexing will be seen in the final part of the problem.

(a) Calculate the first several polynomials (at least up to $i = 5$).

(b) Prove that for $i \geq 4$
$$A_i(q) = A_{i-1}(q) + q^{i-2}A_{i-2}(q), \tag{4}$$
with initial values $A_2(q) = 1$ and $A_3(q) = 1 + q$. Use a combinatorial argument, separating the cases that the largest part is exactly $i - 2$ or something smaller.

(c) Prove that for $i \geq 1$,
$$R_i(q) = R_{i+1}(q) + q^i R_{i+2}(q). \tag{5}$$
There are several possible approaches: directly from (1), or a combinatorial argument that conditions on whether or not the smallest part is i.

(d) Finally, combine the above ideas and prove that for $i \geq 2$,
$$R_1(q) = A_i(q)R_i(q) + q^{i-1}A_{i-1}(q)R_{i+1}(q). \tag{6}$$

Remark: It will be very important later that (6) *follows directly from* (5)*, as an inductive argument also gives the recurrence for the* $A_i(q)$ *without any appeal to the combinatorics of the gap-2 condition. Parts (a) and (b) were included only for the sake of further exploring/motivating these combinatorial properties of Rogers-Ramanujan partitions.*

2. (a) Prove that the $\lim_{i\to\infty} A_i(q)$ exists, so that it makes sense to write $A_\infty(q)$. You may argue either with formal power series, explaining why the coefficients "stabilize" as $i \to \infty$; or analytically (for $|q| < 1$), for example by using dominated convergence and comparing to $P(q)$; or try to do both!

(b) Prove that $\lim_{i\to\infty} R_i(q) = 1$. Again, you may argue as formal power series, analytically, or both.

(c) Now use (6) and conclude that $R_1(q) = A_\infty(q)$.

3. Now consider the product sides of (2) – (3). Define
$$G_1(q) := \frac{1}{(q, q^4; q^5)_\infty}, \qquad \text{and} \qquad G_2(q) := \frac{1}{(q^2, q^3; q^5)_\infty}.$$

(a) Show that
$$G_1(q) = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty},$$
and then apply the Jacobi Triple Product to conclude that
$$G_1(q) = \frac{1}{(q;q)_\infty}\left[1 + \sum_{n\geq 0}(-1)^{n+1}q^{\frac{5n^2+9n+4}{2}}\left(1 + q^{n+1}\right)\right].$$

(b) Similarly, show that
$$G_2(q) = \frac{1}{(q;q)_\infty}\left[1 + \sum_{n\geq 0}(-1)^{n+1}q^{\frac{5n^2+7n+2}{2}}\left(1 + q^{3n+3}\right)\right]. \tag{7}$$

(c) Combine the above and verify that

$$G_1(q) - G_2(q) = \frac{1}{(q;q)_\infty} \cdot q \left[\sum_{n\geq 0} (-1)^n q^{\frac{5n^2+7n}{2}} \left(1 - q^{n+1}\right)\left(1 - q^{2n+2}\right) \right]. \tag{8}$$

4. For $i \geq 2$, define the series

$$g_i(q) := \sum_{n\geq 0} (-1)^n q^{\frac{5n^2-5n}{2}+2in} \left(q^{n+1};q\right)_{i-2} \left(1 - q^{2n+i-1}\right). \tag{9}$$

Note that $g_2(q) = (q;q)_\infty G_2(q)$ (it requires a short calculation to rewrite (7)), and (8) implies that $g_3(q) = (q;q)_\infty \cdot q\left(G_1(q) - G_2(q)\right)$.

(a) Separate the $n = 0$ term and shift the remaining sum to obtain

$$g_i(q) = (q;q)_{i-2}\left(1 - q^{i-1}\right) + \sum_{n\geq 0} (-1)^{n+1} q^{\frac{5n^2+5n}{2}+2i(n+1)} \left(q^{n+2};q\right)_{i-2} \left(1 - q^{2n+i+1}\right).$$

(b) Now split the term $\left(1 - q^{2n+i}\right)$ to obtain

$$\begin{aligned} g_{i+1}(q) =& \sum_{n\geq 0} (-1)^n q^{\frac{5n^2-5n}{2}+2(i+1)n} \left(q^{n+1};q\right)_{i-1} \\ &+ \sum_{n\geq 0} (-1)^{n+1} q^{\frac{5n^2-5n}{2}+2(i+1)n+2n+i} \left(q^{n+1};q\right)_{i-1}. \end{aligned}$$

In the first sum above, isolate the $n = 0$ term, and shift the remaining sum, and conclude

$$\begin{aligned} g_{i+1}(q) =& (q;q)_{i-1} + \sum_{n\geq 0} (-1)^{n+1} q^{\frac{5n^2+5n}{2}+2(i+1)(n+1)} \left(q^{n+2};q\right)_{i-1} \\ &+ \sum_{n\geq 0} (-1)^{n+1} q^{\frac{5n^2-5n}{2}+2(i+1)n+2n+i} \left(q^{n+1};q\right)_{i-1}. \end{aligned}$$

(c) Now use (a) and (b) to show that

$$g_i(q) - g_{i+1}(q) = q^i g_{i+2}(q). \tag{10}$$

After (carefully!) grouping terms, you should find that you need an identity of the form

$$1 - q^a - q^b + q^{2a+b} + q^{a+2b} - q^{2a+2b} = (1 - q^a)(1 - q^b)(1 - q^{a+b}).$$

5. (a) Using (9), prove that $\lim_{i\to\infty} g_i(q) = (q;q)_\infty$.

(b) For $i \geq 2$, define $G_i(q) := (q;q)_\infty^{-1} g_i(q)$. Conclude that $\lim_{i\to\infty} G_i(q) = 1$.

(c) Referring to (10) (you may assume its truth if you did not answer the problem), show that for $i \geq 1$,

$$G_i(q) = G_{i+1}(q) + q^i G_{i+2}(q). \tag{11}$$

As mentioned in the remark following 1, it is now possible to iterate (11) and conclude that for $i \geq 2$,

$$G_1(q) = A_i(q)G_i(q) + q^{i-1}A_{i-1}(q)G_{i+1}(q). \tag{12}$$

Work out the details, showing that this is compatible with the definition of A_i above.

(d) The proof of the first Rogers-Ramanujan identity (2) now concludes by taking the limit as $i \to \infty$ in (12), which implies that $G_1(q) = A_\infty(q)$. Comparing to Problem 2, you have now proven $G_1(q) = A_\infty(q) = R_1(q)$!

6. The second Rogers-Ramanujan identity (3) follows by arguments nearly identical to Problems 1 – 5, so you will only need to explain a few differences as described here.

(a) Define the polynomials $B_i(q)$ by

$$B_i(q) = B_{i-1}(q) + q^{i-2}B_{i-2}(q)$$

for $i \geq 4$, and $B_2 = B_3 = 1$. Prove that for $i \geq 3$,

$$R_2(q) = B_i(q)R_i(q) + q^{i-1}B_{i-1}(q)R_{i+1}(q).$$

Remark: Although a combinatorial argument is not required, one approach uses the observation that

$$B_i(q) = \sum_{\substack{\lambda \in \mathcal{RR}_{\geq 2} \\ \lambda_1 \leq i-2}} q^{|\lambda|}.$$

(b) Starting from (11), show (analogously to (12)) that

$$G_2(q) = B_i(q)G_i(q) + q^{i-1}B_{i-1}(q)G_{i+1}(q).$$

(c) Now conclude that $R_2(q) = B_\infty(q) = G_2(q)$.

In Problems 7 – 8 you will explore some of the ideas behind Dyson's partition statistics, which were introduced in order to provide a combinatorial decomposition of the Ramanujan congruences. As discussed in lecture, he defined the *rank* of a partition to be the largest part minus the number of parts, so

$$\text{rank}(\lambda) := \alpha(\lambda) - \ell(\lambda).$$

Define the enumeration function

$$N(m, n) := \#\{\lambda \in \mathcal{P} \mid \lambda \vdash n, \ \text{rank}(\lambda) = m\}.$$

7. (a) Prove that if $\lambda \vdash n$, then the possible range of the rank function is given by $-(n-1) \leq \text{rank}(\lambda) \leq n - 1$. Do all values in this range occur?

(b) Prove that the rank function is symmetric between its negative and positive values; in other words, $N(m,n) = N(-m,n)$.

(c) Dyson's observation was that if the rank is reduced modulo 5, then the partitions of $5n+4$ are divided into 5 equinumerous classes. In particular, define

$$N(m,k;n) := \#\{\lambda \in \mathcal{P} \mid \lambda \vdash n, \ \mathrm{rank}(\lambda) \equiv m \pmod{k}\},$$

and verify that $N(m,k;n) = \sum_{r\in\mathbb{Z}} N(m+kr,n)$. Dyson's claim is then written as

$$N(m,5;5n+4) = \frac{1}{5}p(5n+4), \quad 0 \le m \le 4;$$
$$N(m,7;7n+5) = \frac{1}{7}p(7n+5), \quad 0 \le m \le 6.$$

Verify the claim for the 30 partitions of 9.

(*Optional*) Calculate the rank modulo 5 for all partitions of 14, and calculate the rank modulo 7 for all partitions of 12. You probably should not do all of these by hand - use computational software!

8. Recall that earlier in the semester we defined the generating function

$$f(x,y) = f(x,y;q) := \sum_{\lambda\in\mathcal{P}} x^{\ell(\lambda)} y^{\alpha(\lambda)} q^{|\lambda|},$$

and used combinatorial arguments (condititioning on the largest part) to show

$$f(x,y) = 1 + \sum_{n\ge 1} \frac{xy^n q^n}{(xq;q)_n}. \tag{13}$$

We then provided two proofs that $f(x,y) = f(y,x)$, using combinatorial arguments (based on conjugation) and analytic arguments (using Heine's ${}_2\phi_1$ transformation). Note that the symmetry in x and y is **not** apparent in (13).

(a) Prove that the generating function for the rank can be expressed in terms of f, specifically as

$$R(u;q) := \sum_{\lambda\in\mathcal{P}} u^{\mathrm{rank}(\lambda)} q^{|\lambda|} = f\left(u, u^{-1}; q\right). \tag{14}$$

(b) In the next two parts you will prove the alternative representation

$$f(x,y) = \sum_{n\ge 0} \frac{x^n y^n q^{n^2}}{(xq, yq; q)_n}. \tag{15}$$

First, give a combinatorial argument using Durfee squares. Recall that if the (largest) Durfee square of size n is removed from a partition, then the remaining parts to the right form a conjugate partition with parts $\le n$, which only contribute to the largest part. The remaining parts below the square form a partition with parts $\le n$, which only contribute to the number of parts.

(c) Recall the notation for basic hypergeometric series:

$$ {}_3\phi_2\left(\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q, t\right) := \sum_{n\geq 0} \frac{(a,b,c;q)_n t^n}{(q,d,e;q)_n}. $$

For an analytic proof of (15), use the following transformation formula ((3.2.7) in Gaspar-Rahman):

$$ {}_3\phi_2\left(\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q, \frac{de}{abc}\right) = \frac{\left(\frac{e}{a}, \frac{de}{bc}; q\right)_\infty}{\left(e, \frac{de}{abc}; q\right)_\infty} \cdot {}_3\phi_2\left(\begin{matrix} a, & \frac{d}{b}, & \frac{d}{c} \\ & d, & \frac{de}{bc} \end{matrix}; q, \frac{e}{a}\right). \tag{16} $$

Make the substitutions $a = q, d = xq, e = yq$ and $b, c \to 0$.

9. In 1929 Watson proved an analog to Whipple's ${}_7F_6$ transformation, which you may use without proof in this problem (although Watson's proof is only 3 pages long, and the heaviest machinery it requires is Heine's ${}_2\phi_1$ transformation). In particular, if $|q| < 1$ and $g = q^{-N}$ for some $N \geq 0$ (all other parameters may be arbitrary complex values), then

$$ {}_8\phi_7\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & c, & d, & e, & f, & g \\ & \sqrt{a}, & -\sqrt{a}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & \frac{aq}{f}, & \frac{aq}{g} \end{matrix}; q, \frac{a^2q^2}{cdefg}\right) $$

$$ = \frac{\left(aq, \frac{aq}{fg}, \frac{aq}{ge}, \frac{aq}{ef}; q\right)_\infty}{\left(\frac{aq}{e}, \frac{aq}{f}, \frac{aq}{g}, \frac{aq}{efg}; q\right)_\infty} \cdot {}_4\phi_3\left(\begin{matrix} \frac{aq}{cd}, & e, & f, & g \\ & \frac{efg}{a}, & \frac{aq}{c}, & \frac{aq}{d} \end{matrix}; q, q\right). $$

This is now known as the *Watson-Whipple* transformation. Note that it is a **finite** summation (with N terms) on the left side, and furthermore, the product on the right is actually a rational function. However, an infinite summation can be obtained by taking the limit $N \to \infty$, which is equivalent to setting $g \to \infty$.

(a) Show that

$$ \frac{(q\sqrt{a}, -q\sqrt{a}; q)_n}{(\sqrt{a}, -\sqrt{a}; q)_n} = \frac{1 - aq^{2n}}{1 - a}. $$

(b) Set $c, d, e, f, g, \mapsto \infty$ in Watson-Whipple, and simplify the result. You should obtain the identity

$$ 1 + \sum_{n\geq 1} \frac{(aq;q)_{n-1}}{(q;q)_n} a^{2n} q^{\frac{5n^2-n}{2}} \left(1 - aq^{2n}\right) = (aq;q)_\infty \sum_{n\geq 0} \frac{a^n q^{n^2}}{(q;q)_n}. $$

(c) The first Rogers-Ramanujan identity (2) now follows by setting $a = 1$ and using the Jacobi Triple Product.

(d) To obtain the second Rogers-Ramanujan identity (3), set $a = q$; it now requires slightly more work to simplify.