MATH 7230 Homework 9 - Spring 2017

Due Thursday, Apr. 27 at 10:30

www.math.lsu.edu/∼mahlburg/

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Andrews A.B" means Example B at the end of Chapter A in the textbook.

In Problems 1 – 2 you will prove additional properties for the partition rank function. Recall from Homework 8 that

$$\mathrm{rank}(\lambda) := \alpha(\lambda) - \ell(\lambda).$$

1. On Homework 8 Problem 8 you showed the generating function identity

$$R(u;q) := \sum_{\lambda \in \mathcal{P}} u^{\mathrm{rank}(\lambda)} q^{|\lambda|} = \sum_{n \geq 0} \frac{q^{n^2}}{(uq, u^{-1}q; q)_n}.$$

   In this problem you will use the Watson-Whipple transformation (see Homework 8 Problem 9) to obtain another representation for $R(u;q)$. The transformation states that if $g = q^{-N}$ for some $N \geq 0$,

$$\begin{aligned} {}_8\phi_7 &\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & c, & d, & e, & f, & g \\ & \sqrt{a}, & -\sqrt{a}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & \frac{aq}{f}, & \frac{aq}{g} \end{matrix} ; q, \frac{a^2q^2}{cdefg} \right) \\ &= \frac{\left(aq, \frac{aq}{fg}, \frac{aq}{ge}, \frac{aq}{ef}; q\right)_\infty}{\left(\frac{aq}{e}, \frac{aq}{f}, \frac{aq}{g}, \frac{aq}{efg}; q\right)_\infty} \cdot {}_4\phi_3 \left(\begin{matrix} \frac{aq}{cd}, & e, & f, & g \\ & \frac{efg}{a}, & \frac{aq}{c}, & \frac{aq}{d} \end{matrix} ; q, q \right). \end{aligned}$$

   (a) Set $e, f, g \to \infty$ in the Watson-Whipple transformation and simplify as much as possible.

   (b) Find the appropriate substitutions for a, b, and c in order to prove (as Watson did in his paper "The Final Problem: An Account of the Mock Theta Functions") that

$$R(u;q) = \frac{1-u}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1 - uq^n}. \tag{1}$$

2. If you did not solve Problem 1, you may assume the formula (1) without proof.

   (a) Verify that $R(1;q) = \dfrac{1}{(q;q)_\infty} = P(q)$, the partition generating function.

   (b) Prove that $R(u;q) = R(u^{-1};q)$ directly from (1) by setting $n \mapsto -n$ in the summation.

Remark: $R(-1;q)$ is Ramanujan's "3rd order mock theta function" $f(q)$, and Bringmann and Ono showed that $R(\zeta;q)$ is a mock theta function (in the modern sense) for all non-trivial roots of unity ζ!

In Problems 3 – 5 you will prove rough bounds for the partition function. For two formal power series with real coefficients, say $f(q) = \sum_{n\geq 0} a(n)q^n$ and $g(q) = \sum_{n\geq 0} b(n)q^n$, define the *strong inequality* relation by

$$f(q) \preceq g(q) \quad \Leftrightarrow \quad a(n) \leq b(n) \text{ for } n \geq 0,$$

with the strict relation $\prec$ defined analogously. For example, $f(q) \succ 0$ means that all coefficients of f are positive. A series $f(q)$ is *monotonically increasing* if $0 \leq a(0) \leq a(1) \leq a(2) \leq \cdots$.

3. If $f(q) = \sum_{n\geq 0} a(n)q^n$, the *difference* series is defined to be

$$f_d(q) := a(0) + \sum_{n\geq 1} (a(n) - a(n-1))q^n.$$

   (a) Show that $f_d(q) = (1-q) \cdot f(q)$.

   (b) Prove that $f(q)$ is monotonically increasing if and only if $f_d(q) \succeq 0$.

4. In this problem you will prove that $p(n)$ is *super-polynomial.* This means that for any $k \geq 0$, there is an $N > 0$ and a constant c such that $p(n) > cn^k$ for $n \geq N$.

   (a) Prove that if $f_i(q) \succeq h_i(q)$ for $i = 1, 2$, then

$$f_1(q) \cdot f_2(q) \succeq h_1(q) \cdot h_2(q).$$

   (b) Show that for any integer $k \geq 1$,

$$P(q) \succeq \frac{1}{(1-q^{k!})^k}.$$

   Hint: First show that if $d \mid n$, then $\frac{1}{1-q^d} \succeq \frac{1}{1-q^n}$. Now use part (a).

   (c) Prove that $P(q)$ is monotonically increasing. There are several possible approaches:

   - (Combinatorial) Let $\mathcal{P}(n)$ denote the partitions of n, and find an injection from $\mathcal{P}(n-1)$ to $\mathcal{P}(n)$;
   - (Generating functions) Consider $(1-q) \cdot P(q)$ and use Problem 3;
   - (Analytic) Calculate $p(n)$ as the n-th Taylor coefficient (around $q = 0$) of $P(q)$; this gets messy!

   (d) Finish the proof by applying the Binomial Theorem to part (b), which will give you a lower bound for $p(mk!)$ for all m. Then use (c) to obtain a lower bound for all other n.

Remark: The constants that you find using this argument are quite small, involving $\frac{1}{k!}$, but this is sufficient.

5. Now you will show various exponential upper bounds. We say that $p(n)$ is at most *exponential of order* α if there is $N > 0$ and a constant c such that $p(n) \leq c\alpha^n$ for $n \geq N$. The strongest claim is that $p(n)$ is at most exponential of order α for all $\alpha > 1$.

   (a) Recall the formula for the composition function $c(n)$ from Homework 1. Describe the relationship between $p(n)$ and $c(n)$, and conclude that $p(n)$ is at most exponential of order $\alpha = 2$.

   (b) Now improve this to $\alpha = \phi = \frac{1+\sqrt{5}}{2}$ by showing that $p(n) \leq F_{n+1}$ (the Fibonacci sequence, defined by $F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n$). In particular, define a map $\sigma : \mathcal{P}(n) \to \mathcal{P}(n-1) \cup \mathcal{P}(n-2)$ (note that this is a disjoint union) as follows:

$$\sigma(\lambda) := \begin{cases} \big(\lambda_1, \ldots, \lambda_{\ell(\lambda)-1}\big) & \text{if } \lambda_\ell = 1; \\ (n-2) & \text{if } \lambda = (n); \\ (\lambda_1 + \ell(\lambda) - 3, \lambda_2 - 1, \lambda_\ell - 1) & \text{if } \lambda_\ell > 1. \end{cases}$$

   Prove that σ is an injection, and conclude that $p(n) \leq p(n-1) + p(n-2)$, with strict inequality for $n \geq 5$.

   Remark: Alternatively, recall that compositions into 1s and 2s satisfy $c_{12}(n) = F_{n+1}$*, and find an injection from* $\mathcal{P}(n)$ *– I know at least one simple map, and there are probably others!*

   (c) Finally, use the fact that the radius of convergence of $P(q)$ is 1 (Homework 2 Problem 5) and conclude that $p(n)$ is at most exponential of order α for any $\alpha > 1$.

   Hint: For a proof by contradiction, note that if $p(n)$ *is not of order* α*, then there are infinitely many* n *such that* $p(n) \geq c\alpha_n$*. Plug in* $q = \frac{1}{\alpha}$ *and show* $P(q)$ *diverges.*

6. In class we proved that as $s \to 0^+$,

$$\log\big(P(e^{-s})\big) \sim \frac{\pi^2}{6s}.$$

   (a) Let $f_{\mathcal{D}}(q) = \sum_{n \geq 0} Q(n)q^n = (-q;q)_\infty$ be the generating function for partitions into distinct parts. Prove that

$$\log\big(f_{\mathcal{D}}(e^{-s})\big) \sim \frac{\pi^2}{12s}.$$

   (b) Prove the same asymptotic formula for partitions into odd parts, $f_{\mathcal{O}}(q) = \dfrac{1}{(q;q^2)_\infty}$. Although we know by Euler's theorem that the two series are equal, you should do this **directly** from the two different product representations.

   Remark: Hardy and Ramanujan also proved the asymptotic formula

$$Q(n) \sim \frac{1}{4 \cdot 3^{\frac{1}{4}} n^{\frac{1}{4}}} e^{\pi\sqrt{\frac{n}{3}}}.$$

7. This problem is another example of how to use q-series in asymptotic analysis. Let $f(q) := \sum_{n \geq 0} (-1) q^{n^2}$.

(a) Apply the Jacobi Triple Product Identity to prove that

$$f(q) = \frac{1}{2} + \frac{1}{2} \cdot \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.$$

(b) Conclude that

$$\lim_{q \to 1^-} f(q) = \frac{1}{2}.$$

This takes a bit more care than it might appear; note that both the numerator and denominator go to zero as q goes to $1 \ldots$.