# MATH 7230 Homework 1 - Spring 2017

Due Thursday, Jan. 25 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

In Problems 1 – 4 you will explore some lesser-known proofs of the infinitude of the primes.

1. (a) Fill in the details of the following proof that there are infinitely many primes:

   Pick an arbitrary $1 \neq N \in \mathbb{N}$ and set $a(1) := N$. Define

   $$a(2) := N \cdot (N+1),$$
   $$a(3) := N \cdot (N+1) \cdot \big[N \cdot (N+1) + 1\big],$$

   and so on, with $a(m) := a(m-1) \cdot (a(m-1)+1)$. Prove that for all $m$, $a(m)$ has a prime factor $p_m$ that does not divide $a(1), \ldots, a(m-1)$. Conclude that there is an infinite list of distinct primes $p_1, p_2, \ldots$.

   (b) Generalize the above proof by letting $f : \mathbb{N} \to \mathbb{N}$ be any arithmetic function such that $n$ is always coprime to $f(n)$, and constructing the iterative sequence $a(m) := a(m-1) \cdot f(a(m-1))$. For example, in part (a), $f(n) = n+1$.

2. The *Fermat numbers* are defined by $F_n := 2^{2^n} + 1$ for $n \in \mathbb{N}_0$.

   (a) Prove that $F_n = F_0 F_1 \cdots F_{n-1} + 2$.
   *Hint: Try to factor $F_n - 2$.*

   (b) Conclude that all $F_n$ are relatively prime, and therefore, there are infinitely many primes!

3. In this problem you will modify Euclid's proof to obtain special cases of Dirichlet's Theorem on primes in arithmetic progressions.

   (a) First, prove that there are infinitely many primes of the form $4n+3$ as follows: Given a list $q_1, q_2, \ldots, q_k$ of primes of this form, let $N := 4q_1 \cdots q_k - 1$. Prove that $N$ has a prime divisor $q \equiv 3 \pmod 4$ that is not one of the $q_j$s.

   (b) Generalize your result from part (a) as much as possible.
   *Hint: If $p$ is prime and $m \in \mathbb{N}$, what are the possible residues $p \bmod m$?*

4. All of the problems above use only basic properties of arithmetic. In this problem you will also need some basic analysis (in the form of inequalities), while also assuming the Fundamental Theorem of Arithmetic. Suppose that by way of contradiction there are only finitely many primes $p_1, \ldots, p_r$ and choose a large integer $N$.

(a) Prove that each integer $x$ can be factored uniquely as $x = m^2 s$, where $s$ is *square-free*: this means that there is no $d > 1$ such that $d^2 \mid s$.

(b) Now consider all $x \leq N$, and show that the factorization must satisfy $m \leq \sqrt{N}$.

(c) Finally, show that there at most $2^r \sqrt{N}$ distinct factorizations. Complete the proof by choosing $N$ large enough to obtain a contradiction.

5. In this problem you will prove that $e$ is irrational.

   (a) As a warm-up, prove that $2 < e < 3$ by comparing the Taylor expansion $e - 2 = \sum_{n \geq 2} \frac{1}{n!}$ to the geometric series $\sum_{n \geq 1} \frac{1}{2^n}$.

   (b) Suppose to the contrary that $e = \frac{a}{b} \in \mathbb{Q}$. Find a contradiction by showing that

$$b! \left( e - \sum_{n=0}^{b} \frac{1}{n!} \right)$$

   is an integer between 0 and 1.

   *Remark: Actually, it is easier to consider $e^{-1}$, which was suggested by Pennisi (1953).*

(Optional) To learn why $e$ is transcendental, I recommend starting with Hurwitz's proof, which is Theorem 204 in Hardy and Wright. You can also find the argument outlined online at: `http://planetmath.org/eistranscendental`.

An alternative proof follows directly from Euler's continued fraction representation $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \ldots]$, though this requires the theory of continued fraction convergents and Bessel functions.

6. The *Prime Number Theorem* (PNT) states that as $X \to \infty$,

$$\pi(X) := \#\{p \leq X \mid p \text{ prime}\} \sim \frac{X}{\log(X)}.$$

   (a) Using your favorite computational software package (learn one if you don't know any!), calculate $\pi(200), \pi(2000)$, and $\pi(20000)$, and compare values.

   (b) Suppose that the primes are listed in order, as $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. Prove that PNT is equivalent to the following asymptotic formula for the $k$-th prime:

$$p_k \sim k \log(k).$$

   (c) What (if anything) can you now conclude about successive prime gaps, $p_{k+1} - p_k$?

In Problems 7 – 9 you will learn some basic facts about an extremely important function in Number Theory.

7. The *Riemann Zeta function* is defined by

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$

   (a) Prove that $\zeta(s)$ converges absolutely if $\text{Re}(s) > 1$.

(b) Euler's product expansion states that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}. \tag{1}$$

Use the Fundamental Theorem of Arithmetic to prove this as a *formal* identity (i.e., ignoring all questions of convergence, show that the individual terms of the Dirichlet series are the same).

(c) In order to prove that infinite products converge (or diverge), it is often convenient to compare to a sum. First, prove that $1 - x \leq e^{-x}$ for all real $x$. Then prove that

$$1 - x > e^{-x-x^2}$$

for $0 < x < \frac{1}{2}$.

(d) Use parts (b) and (c) to prove that (1) also converges for $\text{Re}(s) > 1$. You need to show that $\prod \left(1 - \frac{1}{p^s}\right)$ is **not** zero.

8. (a) Prove that $\zeta(1)$ diverges to $+\infty$. Try to provide multiple arguments: Series tests; Integral Comparisons; etc.

   (b) Recalling (1), give another proof that there are infinitely many primes.

9. It is an amazing fact that $\zeta(2) = \frac{\pi^2}{6}$ (this is known as "Basel's problem", and the first proof is generally attributed to Euler. It is also a fact that $\pi^2$ (and indeed, any rational power of $\pi$) is irrational.

   Use these facts to provide yet another (very short!) proof that there are infinitely many primes.

   *Remark: We will revisit Euler's original proof using infinite product expansions later in the semester. The more standard "modern" proofs generally use Fourier series.*