MATH 7230 Homework 10 - Spring 2017

Due Friday, May 4 at 5:00

Website: `www.math.lsu.edu/~mahlburg/teaching/2018-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

Problems 1–4 address the use of the Hyperbola Method for calculating the average order of the generalized divisor function

$$d_k(n) := \sum_{\substack{m_1,\ldots,m_k \in \mathbb{N} \\ m_1 \cdots m_k = n}} 1. \tag{1}$$

Recall that we have previously seen that for $d_2(n) (= \sigma_0(n))$ the average order is given by MV Theorem 2.3:

$$\sum_{n \leq X} d_2(n) = X \log X + (2\gamma - 1)X + O\left(\sqrt{X}\right).$$

You will prove an analogous formula for general $k \geq 2$,

$$D_k(X) := \sum_{n \leq X} d_k(n) = X P_{k-1}(\log X) + O\left(X^{1-\frac{1}{k}} (\log X)^{k-2}\right), \tag{2}$$

where $P_j(y) = \dfrac{y^j}{j!} + \cdots \in \mathbb{R}[y]$. Note that the case $k = 1$ also nearly has this shape, but with an error term of $O(1)$ rather than $O((\log X)^{-1})$.

1. (a) Find $\liminf\limits_{n \to \infty} d_k(n)$.

   (b) We saw previously that the "largest" values of $d_2(n)$ satisfy

   $$\log(d_2(n)) \sim \frac{\log 2 \log n}{\log \log n},$$

   so that (up to a multiplicative constant) $d_2(n) \sim n^{\log 2 / \log \log n}$ (see MV Theorem 2.11 and Homework 4 #9).

   Prove that for all k and n, $d_k(n) \leq n^{k-1}$.

2. Recall that the definition of $d_k(n)$ is equivalent to the k-fold convolution of the constant function 1, i.e. $d_k = \underbrace{1 \star 1 \star \cdots \star 1}_{k}$. Following a suggestion from T. Tao's lecture notes, write $1 = \underline{1} + \overline{1}$, where

$$\underline{1}(n) := \begin{cases} 1 & \text{if } n \leq X^{\frac{1}{3}}, \\ 0 & \text{if } n > X^{\frac{1}{3}}; \end{cases} \qquad \overline{1}(n) := \begin{cases} 0 & \text{if } n \leq X^{\frac{1}{3}}, \\ 1 & \text{if } n > X^{\frac{1}{3}}. \end{cases}$$

(a) Explain why plugging in gives

$$d_3 = \underline{1} \star \underline{1} \star \underline{1} + 3\left(\underline{1} \star \underline{1} \star \overline{1}\right) + 3\left(\underline{1} \star \overline{1} \star \overline{1}\right) + \overline{1} \star \overline{1} \star \overline{1}.$$

In particular, what commutativity property is needed to group terms?

(b) Unfortunately, the third term above is not easily manageable as written. Instead, the following alternative representation is more helpful:

$$d_3 = \underline{1} \star \underline{1} \star \underline{1} - 3\left(\underline{1} \star \underline{1} \star 1\right) + 3\left(\underline{1} \star 1 \star 1\right) + \overline{1} \star \overline{1} \star \overline{1}. \tag{3}$$

Prove this!

Remark: A slick proof uses the binomial expansion $(1-z)^3 = 1 - 3z + 3z^2 - z^3$, *with* $\underline{1}$ *playing the role of* z....

3. In lecture we used (3) to show that

$$D_3(X) = \left\lfloor X^{\frac{1}{3}} \right\rfloor^3 - 3 \sum_{m1, m_2 \le X^{\frac{1}{3}}} D_1\left(\frac{X}{m_1 m_2}\right) + 3 \sum_{m_1 \le X^{\frac{1}{3}}} D_2\left(\frac{X}{m_1}\right).$$

We calculated that the leading asymptotic term is indeed $X\frac{(\log X)^2}{2}$, so $P_2(y) = \frac{y^2}{2} + c_1 y + c_0$. Using partial summation, calculate the remaining coefficients c_1 and c_0, and verify that the error term is $O\left(X^{\frac{2}{3}} \log X\right)$. Be warned, this is **lengthy!**

In particular, you should find that $c_1 = 3\gamma - 1$ and $c_0 = 3\gamma^2 - 3\gamma + 1 - 3\gamma_1$, where $\gamma_1 := \lim_{N\to\infty} \left(\sum_{n \le N} \frac{\log n}{n} - \frac{(\log N)^2}{2} \right)$.

Remark: Using complex analysis, it can be shown that $P_2(\log x)$ *is the residue at* $s = 1$ *of* $\frac{\zeta^3(s) x^s}{s}$.

4. In lecture we used the simplest version of the Hyperbola Method to inductively obtain the main term of (2). For example, the calculation in the case $k = 3$ is

$$\begin{aligned} D_3(X) &= \sum_{m \le X} d_2(m) \left\lfloor \frac{X}{m} \right\rfloor = X \sum_{m \le X} \frac{d_2(m)}{m} + O\left(D_2(X)\right) \\ &= X\left(D_2(X) \cdot \frac{1}{X} + \int_1^X D_2(t) \frac{1}{t^2} dt\right) + O\left(D_2(X)\right) \\ &= \frac{X (\log X)^2}{2} + O(D_2(X)), \end{aligned}$$

using the fact that $\displaystyle\int_1^X D_2(t)\frac{1}{t^2}dt \sim \int_1^X \frac{\log t}{t} dt = \frac{(\log X)^2}{2}$.

However, Problem 2 suggests a way to use the general Hyperbola Method (MV equation (2.9)) in order to obtain a more precise formula for $D_3(X)$.

(a) Prove that for any $1 \le Y \le X$,

$$D_3(X) = \sum_{r \le Y} d_2(r) \left\lfloor \frac{X}{r} \right\rfloor + \sum_{s \le \frac{X}{Y}} D_2\left(\frac{X}{s}\right) - D_2(Y) \left\lfloor \frac{X}{Y} \right\rfloor.$$

(b) Pick $Y = X^{\frac{2}{3}}$, and calculate the main asymptotic term – verify that you again get $X\frac{(\log X)^2}{2}$.

Problems 5 address the first term from Goldston-Pintz-Yildirim's sieve, as described in Granville Section 4.7. Here there are several details to fill in.

5. The sum of interest is

$$S_2 := \sum_{\substack{d_1,d_2 \le R \\ \text{squarefree} \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\omega^*(D)}{\varphi(D)},$$

where all notation is the same as before (see Homework 9). We showed in lecture that

$$S_2 = \sum_{\substack{r \le R \\ \text{squarefree}}} y^*(r)^2 \frac{\omega^*(r)}{\varphi_\omega(r)},$$

where $y^*(r) := \frac{Y^*(r)\varphi_\omega(r)}{\omega^*(r)}$,

$$Y^*(r) := \mu(r) \sum_{\substack{m\,:\,r|m \\ \text{squarefree}}} L^*(m)$$

and finally, $L^*(d) := \lambda(d)\frac{\omega^*(d)}{\varphi(d)}$ (so all of these functions are only supported on squarefree values up to R).

Plug in and simplify in order to prove that

$$y^*(r) = \frac{r}{\varphi(r)} \sum_{m\,:\,(m,r)=1} \frac{y(mr)}{\varphi(n)}.$$

6. Given a fixed r, prove that as $R \to \infty$

$$\sum_{\substack{m \le R \\ \text{squarefree} \\ (m,r)=1}} \frac{1}{\varphi(m)} \sim \frac{\varphi(r)}{r} \log R.$$

Hint: One approach is to use an appropriately defined Dirichlet series, as in Homework 7 Problems 1–4. This was also discussed in lecture.

Remark: See MV 2.1.17 for the case $r = 1$, and MV 2.1.13 for a related problem (without the squarefree restriction).

7. In this problem you will complete the final technical point where S_2 differs from S_1 by reducing $y^*(r)$ to an integral.

(a) Assume that $y(t) = F\left(\frac{\log t}{\log R}\right)$ where $F(y)$ is supported on $(0, 1)$. Use Problem 6 and partial summation to conclude that

$$y^*(r) = \frac{r}{\varphi(r)} \sum_{\substack{m \le \frac{R}{r} \\ (m,r)=1}} \frac{y(mr)}{\varphi(m)} \sim \int_1^{\frac{R}{r}} F\left(\frac{\log rt}{\log R}\right) \frac{dt}{t}.$$

(b) Make the change of variables $u = \frac{\log rt}{\log R}$ and conclude that this is

$$\log R \int_{\frac{\log r}{\log R}}^{1} F(u) du.$$

Remark: The overall expression for the main term of S_2 requires partial summation once again, using the fact (proved in lecture) that

$$\sum_{r \le R} \frac{\omega^*(r)}{\varphi_\omega(r)} \sim \underbrace{\prod_p \left(1 + \frac{\omega(p)}{\varphi_\omega(p)}\right)\left(1 - \frac{1}{p}\right)^k}_{=: \alpha_c(0) = \kappa(g)} \cdot \frac{(\log R)^{k-1}}{(k-1)!}.$$

One obtains

$$S_2 \sim \alpha_c(0)(\log R)^{k+1} \int_0^1 \left(\int_t^1 F(u) du\right)^2 \frac{t^{k-2}}{(k-2)!} dt.$$